Cubical Categories for Higher-Dimensional Parametricity
Extended Version

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Abstract

We introduce a generalization of cubical sets, which we call *cubical categories*, and use it to develop a framework for higher-dimensional parametricity, all the way up to and including infinity. Our framework has the crucial property that if a model is p-parametric according to our definition, then it is l-parametric for every $l < p$. This is a significant generalization of existing definitions of parametricity, which (apart from our 2018 prequel paper) do not appear to recognize, *e.g.*, our 2018 proof-relevant parametric model, as parametric. We illustrate our framework by constructing a specific p-dimensional cubical category for every $p \in \mathbb{N}$, and giving a corresponding p-parametric model for System F when $p \leq 3$. At dimension 1, these models coincide with Reynolds' parametric model as formulated in our prequel paper, and at dimension 2 with the proof-relevant parametric model introduced there.

1 Introduction

Strachey [30] distinguished between *ad hoc* and parametric polymorphic functions in programming languages, defining a polymorphic program to be *parametric* if it applies the same type-uniform algorithm at each of its type instantiations. Reynolds [25] introduced the notion of *relational parametricity* to model the extensional behavior of parametric programs in System F [13], the formal calculus at the core of polymorphic functional languages. Relationally parametric models capture a key feature of polymorphic programs, namely that they preserve all relations between instantiated types. In other words, in relationally parametric models, polymorphic functions always map related arguments to related results.

In Reynolds' original formulation of parametricity, each System F type is assigned an *object interpretation* as, say, a set or a domain, simultaneously with a *relational interpretation* on that set or domain. This is done by induction on System F's type structure, with object and relational interpretations propagated up the type hierarchy in such a way that two key theorems are implied: an *Identity Extension Lemma*, stating that propagating equality relations on base types up the type hierarchy yields equality relations at more complex types, and an *Abstraction Theorem*, stating that the object interpretation of each System F program is related to itself by the relational interpretation of its type. Variations on these results can also be shown hold for types and terms with any number of free variables. When instantiated judiciously, the Abstraction Theorem can be used to prove, *e.g.*, invariance of polymorphic functions under changes of data representation ([2, 9]), equivalences of programs [15], and so-called "free theorems" which infer properties of programs solely from their types [31].

Since there are no set-theoretic models of System F [26], a meta-theory such as the (extensional) Calculus of Constructions with impredicative Set is a natural choice for expressing Reynolds' original ideas. In this setting, however, the Identity Extension Lemma holds only up to an isomorphism (*e.g.*, the identity type on a product is isomorphic, but not identical, to the product of identity types). In [29] we developed a framework for relational parametricity that naturally accommodates this generality. Moreover, in the process of turning Reynolds' informal construction into a sound model of System F, we were inevitably led to a categorical version of Reynolds' original theory, in which, additionally, types act functorially on isomorphisms and all polymorphic functions respect this action. When using the phrase "Reynolds' model", we will thus mean the version given in [29].

In this paper we use ideas from the theory of cubical sets ([6, 14, 18]) to extend the framework from [29] to *all* dimensions, including infinity. We do so by introducing the novel notions of a *p-dimensional cubical category* and a *p-dimensional cartesian closed cubical category with isomorphisms*. Intuitively, a p-dimensional cubical category associates to each level $l \leq p$ a category of "l-relations." It also tells us how to link the relations at level l to the ones at level $l + 1$, for $l < p$, by means of *face maps*, which project l-relations out of $(l + 1)$-relations, as well as *degeneracies* and *connections*, which turn l-relations into $(l + 1)$-relations.

Formally, a cubical category is a functor from the *cube category* of face maps, denegeracies, and connections to the category of categories internal to some sufficiently structured ambient category. The isomorphisms in such categories must include those up to which the interpretations of type constructors preserve object and relational interpretations. They are carefully propagated throughout our models in such a way that a p-dimensional Identity Extension Lemma is guaranteed to hold. Our main technical result shows that every p-dimensional cartesian closed cubical category with isomorphisms induces a canonical model of the simply typed fragment of System F. Our main definition then defines a model of System F to be *p-parametric* with respect to a given p-dimensional cartesian closed cubical category with isomorphisms if it is compatible, in a precise sense, with this canonical model of simple types. This novel definition not only recognizes all the models discussed here as parametric, but has the crucial property that any model recognized as p-parametric is also recognized as l-parametric if $l < p$. In particular, according to our definition, the proof-relevant model for System F from [29] is indeed 1-parametric as well as 2-parametric. This is a significant generalization of existing definitions of parametricity, since, apart from [29], these do not appear to recognize either Reynolds' model or the proof-relevant parametric model given there as parametric.

Overall, this paper shows that parametricity is not strictly a 1- or even 2-dimensional concept. As evidence that our theory is an appropriate and natural extension of relational parametricity to all higher dimensions, in Section 6 we construct a specific p-dimensional cubical category for every $p \in \mathbb{N}$, and give parametric models corresponding to these categories for $p \leq 3$, which coincide with Reynolds' parametric model for System F at dimension 1, coincide with our proof-relevant model for System F from [29] at dimension 2, and give an entirely new 3-parametric model for System F.

The remainder of this paper is structured as follows. In Section 2 we introduce the higher-dimensional cubical categories that are the ultimate basis for our construction. Section 3 uses these to define cubical functors and natural transformations. Section 4 develops the additional cartesian structure needed to interpret the simply typed fragment of System F, and introduces the cubical subcategories of isomorphisms up to which this structure is to be preserved. Section 5 shows that every category possessed of this structure induces a canonical model of the simply typed fragment of System F, and builds on this observation to define what it means for such a model to be *p-parametric*. Running examples are used to illustrate our ideas in familiar parametric models. Section 6 then illustrates our overall framework by constructing a p-dimensional cubical category for all $p \in \mathbb{N}$ and giving a corresponding p-parametric model for System F for $p \leq 3$. These models coincide with Reynolds' model when $p = 1$ and with the proof-relevant parametric model from [29] when $p = 2$.

Preliminaries: Unless otherwise indicated, our examples will be formulated in an extensional version of the Calculus of Inductive Constructions (eCIC). This type theory has a cumulative hierarchy of universes $\mathbb{U} := \mathbb{U}_0 : \mathbb{U}_1 : \ldots$ where the bottom universe $\mathbb{U}$ is impredicative, dependent products $\Pi_{x:A} B(x)$, dependent sums $\Sigma_{x:A} B(x)$, coproducts $A + B$, finite types, *e.g.*, $\mathsf{0}$, $\mathsf{1}$, and $\mathsf{2}$ (with terms $\top$ and $\bot$), natural numbers $\mathbb{N}$, and extensional identity types $M = N$ supporting the *identity reflection rule*: if $M = N$ is inhabited then $M \equiv N$ definitionally. We use the same symbols $\mathbb{N}$ and $\mathsf{2}$ to refer to the *sets* of natural numbers and booleans $\{\top, \bot\}$, respectively. We use the notation $k : \mathbb{N}$, $\star : \mathsf{2}$ versus $k \in \mathbb{N}$, $\star \in \mathsf{2}$ to indicate whether we are thinking of $\mathbb{N}$ and $\mathsf{2}$ as types or as sets.

Following the standard convention, we represent proof-irrelevant predicates as mappings into the type of *propositions*. A proposition is a type in $\mathbb{U}$ with at most one term; formally, $\mathsf{Prop} := \Sigma_{T:\mathbb{U}} \mathsf{isProp}(T)$ and $\mathsf{isProp}(T) := \Pi_{x,y:T}\, x = y$. The identity reflection rule implies that the identity type $M = N$ is a proposition, and so is the type $\mathsf{isProp}(T)$ for any $T : \mathbb{U}$. We treat any proposition $T : \mathsf{Prop}$ as a type in $\mathbb{U}$.

To construct the cubical hierarchy of higher relations, we use some simple operations on natural numbers. Given $k, l : \mathbb{N}$, the types $k < l$ and $k \leq l$ are defined in the obvious way by induction on $\mathbb{N}$ and are always propositions (either $\mathsf{0}$ or $\mathsf{1}$). As such, the unique witness, if any, is usually left implicit. Ordering on $\mathbb{N}$ is decidable: for any $k, l : \mathbb{N}$ the proposition $(k < l) + (k = l) + (l < k)$ is inhabited, and for any $k, l : \mathbb{N}$ with $k \leq l$ inhabited, the proposition $(k < l) + (k = l)$ is inhabited. Similarly, equality on $\mathsf{2}$ is decidable, *i.e.*, for any $\star_1, \star_2 : \mathsf{2}$, the proposition $(\star_1 = \star_2) + (\star_1 = \overline{\star_2})$ is inhabited, where $\overline{\star}$ denotes the negation of $\star$. We will thus freely use definitions by case analysis on these types.

2 Cubical Categories

In [29] we described how to adapt Reynolds' original approach to relational parametricity to obtain a sound model of System F. According to Reynolds, a closed type T has two interpretations, one as an object $[\![T]\!]_0$ in the category $\mathsf{Rel}(0)$ of sets and one as an object $[\![T]\!]_1$ in the category $\mathsf{Rel}(1)$ of relations, and these must be suitably related. A relation R is represented as a pair $(R_\top, R_\bot)$ of sets together with a proof-irrelevant predicate on pairs $(r_\top, r_\bot) : R_\top \times R_\bot$. So there are two canonical ways of projecting a set out of R, namely $\mathsf{f}_\top(R) := R_\top$ and $\mathsf{f}_\bot(R) := R_\bot$, and we refer to the functors $\mathsf{f}_\top$ and $\mathsf{f}_\bot$ as *face maps*. For a closed type T we require that $\mathsf{f}_\top([\![T]\!]_1) = [\![T]\!]_0 = \mathsf{f}_\bot([\![T]\!]_1)$, which precisely states that $[\![T]\!]_1$ is a binary relation on $[\![T]\!]_0$.

Conversely, given a set A, there is a canonical way of turning it into a relation, namely the equality relation $\mathsf{d}(A)$ on A that relates a_1 and a_2 if and only if the type $a_1 =_A a_2$ is inhabited. We may suppress the subscript A when convenient. We refer to the functor d as a *degeneracy* and require that $[\![T]\!]_1 \cong \mathsf{d}([\![T]\!]_0)$; this makes sense since $\mathsf{f}_\top \circ \mathsf{d} = \mathsf{id} = \mathsf{f}_\bot \circ \mathsf{d}$, so we have a reflexive graph of categories. In other words, the relational interpretation $[\![T]\!]_1$ of a closed type T is, up to isomorphism, just the equality relation on $[\![T]\!]_0$, and this in particular gives us a structural characterization of equality.

In [29] we furthermore showed how, using ideas from Orsanigo *et al.* ([11, 22]), we can turn the aforementioned model into a *proof-relevant* one, in which a relation R on sets A and B can relate elements $a : A$ and $b : B$ in more than one way. This requires us to go one level higher and reason about witnesses of relatedness themselves being suitably related by a yet higher relation. Specifically, given a relation R on A and B, and a relation S on C and D, to relate witnesses $p : R(a, b)$ and $q : S(c, d)$, we should know *a priori* how a relates to c and b to d. This motivates defining a *2-relation* to be a quadruple $(Q_{0\top}, Q_{1\top}, Q_{0\bot}, Q_{1\bot})$ of ordinary relations forming a square

$$\begin{array}{ccc} A & \overset{Q_{0\top}}{\text{———}} & B \\ Q_{1\top}\Big| & & \Big|Q_{1\bot} \\ C & \underset{Q_{0\bot}}{\text{———}} & D \end{array}$$

together with a proof-irrelevant predicate on quadruples $(q_{0\top}, q_{1\top}, q_{0\bot}, q_{1\bot})$ forming a square as follows:

$$\begin{array}{ccc} a : A & \overset{q_{0\top} : Q_{0\top}(a,b)}{\text{—————}} & b : B \\ q_{1\top} : Q_{1\top}(a,c)\Big| & & \Big|q_{1\bot} : Q_{1\bot}(b,d) \\ c : C & \underset{q_{0\bot} : Q_{0\bot}(c,d)}{\text{—————}} & d : D \end{array}$$

So now we have four *face maps* $\mathsf{f}_{0\top}, \mathsf{f}_{1\top}, \mathsf{f}_{0\bot}, \mathsf{f}_{1\bot}$ from the category $\mathsf{Rel}(2)$ of 2-relations down to $\mathsf{Rel}(1)$, one for each edge. We also have four functors in the other direction: given a relation R with domain A and codomain B, we obtain the 2-relation $\mathsf{d}_=(R)$ by placing R on top and bottom, $\mathsf{d}(A)$ on the left, $\mathsf{d}(B)$ on the right, and requiring that $q_{0\top} = q_{0\bot}$; this makes sense since the existence of $q_{1\top}$ and $q_{1\bot}$ guarantees that $a = c$ and $b = d$. The functor $\mathsf{d}_{\|}$ placing R on left and right is entirely analogous. We can also place R diagonally: the 2-relation $\mathsf{c}_\top(R)$ has R on top and left, $\mathsf{d}(B)$ on bottom and right, and requires that $q_{0\top} = q_{1\top}$; the functor $\mathsf{c}_\bot$ placing R on bottom and right is again analogous.

In addition to the equalities $\mathsf{f}_\top \circ \mathsf{d} = \mathsf{id} = \mathsf{f}_\bot \circ \mathsf{d}$ inherited from level 1, we now have many more, *e.g.*, $\mathsf{f}_{1\top} \circ \mathsf{c}_\top = \mathsf{id}$, $\mathsf{f}_{0\top} \circ \mathsf{c}_\bot = \mathsf{d} \circ \mathsf{f}_\top$, and $\mathsf{c}_\top \circ \mathsf{d} = \mathsf{d}_= \circ \mathsf{d}$. Readers familiar with cubical sets will immediately recognize this setup as level 2 of the cubical hierarchy, where $\mathsf{d}_=, \mathsf{d}_{\|}$ are the two *degeneracies* and $\mathsf{c}_\top, \mathsf{c}_\bot$ are the two *connections*. This is case $p = 2$ of the following standard definition:

Definition 1. *The category $\square_p$ for $p \in \mathbb{N} \cup \{\infty\}$ is defined as follows: the objects are natural numbers $l \leq p$, called* levels, *and the morphisms are generated by the arrows below:*

- face maps $\mathsf{f}_l(k, \star) : l + 1 \to l$ *for* $l < p$, $k \leq l$, $\star \in 2$
- degeneracies $\mathsf{d}_l(k) : l \to l + 1$ *for* $l < p$, $k \leq l$
- connections $\mathsf{c}_l(k, \star) : l \to l + 1$ *for* $l < p$, $k < l$, $\star \in 2$

subject to the following relations:

- $\mathsf{f}_l(j, \star_1) \circ \mathsf{f}_{l+1}(i + 1, \star_2) = \mathsf{f}_l(i, \star_2) \circ \mathsf{f}_{l+1}(j, \star_1)$ *if* $j \leq i$
- $\mathsf{f}_l(k, \star) \circ \mathsf{d}_l(k) = \mathsf{id}_l$
- $\mathsf{f}_{l+1}(i, \star) \circ \mathsf{d}_{l+1}(j + 1) = \mathsf{d}_l(j) \circ \mathsf{f}_l(i, \star)$ *if* $i \leq j$
- $\mathsf{f}_{l+1}(i + 1, \star) \circ \mathsf{d}_{l+1}(j) = \mathsf{d}_l(j) \circ \mathsf{f}_l(i, \star)$ *if* $j \leq i$
- $\mathsf{f}_l(k, \star) \circ \mathsf{c}_l(k, \star) = \mathsf{id}_l$
- $\mathsf{f}_l(k + 1, \star) \circ \mathsf{c}_l(k, \star) = \mathsf{id}_l$

- $\mathsf{f}_{l+1}(k,\overline{\star}) \circ \mathsf{c}_{l+1}(k,\star) = \mathsf{d}_l(k) \circ \mathsf{f}_l(k,\overline{\star})$
- $\mathsf{f}_{l+1}(k+1,\overline{\star}) \circ \mathsf{c}_{l+1}(k,\star) = \mathsf{d}_l(k) \circ \mathsf{f}_l(k,\overline{\star})$
- $\mathsf{f}_{l+1}(i,\star_1) \circ \mathsf{c}_{l+1}(j+1,\star_2) = \mathsf{c}_l(j,\star_2) \circ \mathsf{f}_l(i,\star_1)$ *if* $i \leq j$
- $\mathsf{f}_{l+1}(i+1,\star_1) \circ \mathsf{c}_{l+1}(j,\star_2) = \mathsf{c}_l(j,\star_2) \circ \mathsf{f}_l(i,\star_1)$ *if* $j < i$
- $\mathsf{d}_{l+1}(i+1) \circ \mathsf{d}_l(j) = \mathsf{d}_{l+1}(j) \circ \mathsf{d}_l(i)$ *if* $j \leq i$
- $\mathsf{c}_{l+1}(k,\star) \circ \mathsf{d}_l(k) = \mathsf{d}_{l+1}(k) \circ \mathsf{d}_l(k)$
- $\mathsf{c}_{l+1}(j,\star) \circ \mathsf{d}_l(i) = \mathsf{d}_{l+1}(i+1) \circ \mathsf{c}_l(j,\star)$ *if* $j < i$
- $\mathsf{c}_{l+1}(j+1,\star) \circ \mathsf{d}_l(i) = \mathsf{d}_{l+1}(i) \circ \mathsf{c}_l(j,\star)$ *if* $i \leq j$
- $\mathsf{c}_{l+1}(k,\star) \circ \mathsf{c}_l(k,\star) = \mathsf{c}_{l+1}(k+1,\star) \circ \mathsf{c}_l(k,\star)$
- $\mathsf{c}_{l+1}(i,\star_1) \circ \mathsf{c}_l(j,\star_2) = \mathsf{c}_{l+1}(j+1,\star_2) \circ \mathsf{c}_l(i,\star_1)$ *if* $i < j$

In our case, each level $l = 0,1,2$ of $\square_2$ corresponds to a category – of sets, relations, and 2-relations, respectively – and each arrow of $\square_2$ corresponds to a functor. So what we have is a functor from $\square_2$ into the category of small categories internal to an appropriate ambient category $\mathcal{C}$. Formally:

Definition 2. *Let* $p \in \mathbb{N} \cup \{\infty\}$ *and* $\mathcal{C}$ *be a locally small finitely complete category. A* p-dimensional cubical category over $\mathcal{C}$ *is a functor* $\mathcal{X} : \square_p \to \mathsf{Cat}(\mathcal{C})$.

To enhance readability, we will mostly refrain from explicit mentions of the ambient category $\mathcal{C}$ and treat categories C and D, and functors $F : C \to D$, internal to $\mathcal{C}$ as ordinary categories and functors. When needed, we denote by C_0 and C_1 the objects of $\mathcal{C}$ representing the objects and morphisms of C, respectively, and by F_0 and F_1 the arrows of $\mathcal{C}$ representing the object and morphism parts of F, respectively.

Because of the well-known problem with Reynolds' original treatment of relational parametricity – namely, that the universe of sets is not impredicative, thus making it impossible to form his "set" interpretations of $\forall$-types in $R(0)$ [26] – the following two running examples are formulated in eCIC. In this case the ambient category $\mathcal{C}$ has types $A : \mathbb{U}_1$ as objects and triples (A, B, f) with $f : A \to B$ as morphisms; using the higher universe $\mathbb{U}_1$ ensures that $\mathbb{U}$ is itself an object in $\mathcal{C}$. Our previous discussion thus shows the following:

Example 3 (Reynolds' model)**.** *The categories* $\mathsf{Rel}(0)$, $\mathsf{Rel}(1)$ *of sets and relations with the associated face maps* $\mathsf{f}_\top$, $\mathsf{f}_\bot$ *and the degeneracy* d *define a* 1*-dimensional cubical category.*

Example 4 (Proof-relevant Reynolds' model)**.** *The categories* $\mathsf{Rel}(0)$, $\mathsf{Rel}(1)$, $\mathsf{Rel}(2)$ *of sets, relations, and 2-relations with the associated face maps* $\mathsf{f}_\top$, $\mathsf{f}_\bot$, $\mathsf{f}_{0\top}, \mathsf{f}_{1\top}, \mathsf{f}_{0\bot}, \mathsf{f}_{1\bot}$*, the degeneracies* $\mathsf{d}, \mathsf{d}_=, \mathsf{d}_\|$*, and the connections* $\mathsf{c}_\top, \mathsf{c}_\bot$ *define a* 2*-dimensional cubical category.*

There are, of course, other possibilities for $\mathcal{C}$. In [29] we showed how the canonical PER model of [19] arises as an instance of the framework for relational parametricity given there by taking $\mathcal{C}$ to be the category of ω-sets and realizable functions, and in Section 6 we give the term model of System F as an instance of our framework at dimension $p = 0$ over the category $\mathcal{C}$ of System F types and terms.

It is easy to see that any morphism in the image of $\mathcal{X}$ can be factored as a composition of face maps followed by a composition of connections followed by a composition of degeneracies, with each of these (possibly empty) compositions in canonical form. A composition $\mathsf{f}_l(k_1,\star_1) \circ \ldots \circ \mathsf{f}_{l+n}(k_n,\star_n)$ of face maps is in canonical form if $k_1 \geq \ldots \geq k_n$; a composition $\mathsf{d}_{l+n}(k_n,\star_n) \circ \ldots \circ \mathsf{d}_l(k_1,\star_1)$ of degeneracies is in canonical form if $k_n \leq \ldots \leq k_1$; and a composition $\mathsf{c}_{l+n}(k_n,\star_n) \circ \ldots \circ \mathsf{c}_l(k_1,\star_1)$ of connections is in canonical form if $k_n \geq \ldots \geq k_1$ and, moreover, $k_{i+1} = k_i$ implies $\star_{i+1} = \overline{\star_i}$ for every $1 \leq i < n$.

As can be seen from the above definition, we do not assume that the relations are constructed in any specific way, but rather only that the abstract operations on relations – the face maps, degeneracies, and connections – suitably interact. In particular, we allow the case when *all* of the relations we have are *derived*, *i.e.*, expressible as a composition of degeneracies and/or connections; at any level l, there is a unique such l-relation corresponding to an object X in $\mathcal{X}(0)$. While many treatments of parametricity impose extra conditions to rule out this case – and we could likewise require that the functor $\mathcal{X}$ be *faithful*, which would disallow it – we opt not to do so here. Instead, we take the view that if, according to our definition, a model is parametric over such a cubical category, then it still deserves to be called so since it *does* satisfy the Abstraction Theorem and the Identity Extension Lemma, albeit with respect to a very limited notion of a (higher) relation.

3 Cubical Functors and Natural Transformations

A p-dimensional cubical category is thus a family of categories, one for each $l \leq p$, interrelated by face maps, degeneracies, and connections. Correspondingly, p-dimensional cubical functors and natural transformations are families of functors and natural transformations, respectively, whose components suitably respect face maps, degeneracies, and connections.

Definition 5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be p-dimensional cubical categories. A* p-dimensional cubical functor *$\mathcal{F}$ from $\mathcal{X}$ to $\mathcal{Y}$ is a family of functors $\{\mathcal{F}(l) : \mathcal{X}(l) \to \mathcal{Y}(l)\}_{l \leq p}$ that respects face maps, degeneracies, and connections in the following sense:*

- *For any $l, k \in \mathbb{N}$ with $k \leq l < p$ and $\star \in 2$, the following diagram of functors commutes:*

$$\begin{array}{ccc} \mathcal{X}(l+1) & \xrightarrow{\mathcal{F}(l+1)} & \mathcal{Y}(l+1) \\ {\scriptstyle \mathcal{X}(\mathsf{f}_l(k,\star))}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}(\mathsf{f}_l(k,\star))} \\ \mathcal{X}(l) & \xrightarrow[\mathcal{F}(l)]{} & \mathcal{Y}(l) \end{array}$$

- *For any $l, k \in \mathbb{N}$ with $k \leq l < p$, the diagram*

$$\begin{array}{ccc} \mathcal{X}(l) & \xrightarrow{\mathcal{F}(l)} & \mathcal{Y}(l) \\ {\scriptstyle \mathcal{X}(\mathsf{d}_l(k))}\downarrow & \Downarrow \varepsilon_l(k) & \downarrow{\scriptstyle \mathcal{Y}(\mathsf{d}_l(k))} \\ \mathcal{X}(l+1) & \xrightarrow[\mathcal{F}(l+1)]{} & \mathcal{Y}(l+1) \end{array}$$

  commutes up to a chosen natural isomorphism $\varepsilon_l(k)$ that satisfies the coherence conditions in Figure 1, corresponding to the three cubical equations determining the faces of a degeneracy.

- *For any $l, k \in \mathbb{N}$ with $k < l < p$ and $\star \in 2$, the diagram*

$$\begin{array}{ccc} \mathcal{X}(l) & \xrightarrow{\mathcal{F}(l)} & \mathcal{Y}(l) \\ {\scriptstyle \mathcal{X}(\mathsf{c}_l(k,\star))}\downarrow & \Downarrow \upsilon_l(k,\star) & \downarrow{\scriptstyle \mathcal{Y}(\mathsf{c}_l(k,\star))} \\ \mathcal{X}(l+1) & \xrightarrow[\mathcal{F}(l+1)]{} & \mathcal{Y}(l+1) \end{array}$$

  commutes up to a chosen natural isomorphism $\upsilon_l(k,\star)$ that satisfies the coherence conditions in Figure 2, corresponding to the six cubical equations determining the faces of a connection.

We further require that the coherence conditions in Figure 3, corresponding to the six cubical equations governing the exchange of degeneracies and/or connections, likewise hold.

The families ε and υ of natural isomorphisms are part of the definition of a cubical functor, which is therefore a triple $(\mathcal{F}, \varepsilon, \upsilon)$. In the case when all $\varepsilon_l(k)$ and $\upsilon_l(k,\star)$ are in fact identities – and thus all three diagrams above commute on the nose – we say that $\mathcal{F}$ is strict.

We may write $(\mathcal{F}, \varepsilon, \upsilon) : \mathcal{X} \to \mathcal{Y}$, or even $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ if ε and υ are unimportant or clear from context, when $(\mathcal{F}, \varepsilon, \upsilon)$ is a cubical functor from $\mathcal{X}$ to $\mathcal{Y}$. As Definition 5 spells out, face maps are respected on the nose, whereas degeneracies and connections are required to be preserved only up to a coherent natural isomorphism. This mirrors the situation in Reynolds' approach. There, for a closed type T, we require strict commutativity with face maps, *i.e.*, $\mathsf{f}_\top(\llbracket T \rrbracket_1) = \llbracket T \rrbracket_0 = \mathsf{f}_\bot(\llbracket T \rrbracket_1)$. By contrast, commutativity with the degeneracy – *i.e.*, $\llbracket T \rrbracket_1 \cong \mathsf{d}(\llbracket T \rrbracket_0)$ – need only hold up to isomorphism.

The reason for a strict treatment of face maps and lax treatment of degeneracies and connections becomes clear when we examine our running examples. Let $\mathsf{Rel} : \square_1 \to \mathsf{Cat}$ be the 1-dimensional cubical category from Example 3, and suppose we want to construct a cubical functor $\mathcal{F} : \mathsf{Rel} \to \mathsf{Rel}$. If R is a relation on A and B, *i.e.*, the carrier of R is the pair (A, B) of sets, then to define the action of $\mathcal{F}(1)$ on R we first need to specify the carrier of the intended relation $\mathcal{F}(1)\,R$. One (indeed, the only) reasonable candidate is the pair $(\mathcal{F}(0)\,A, \mathcal{F}(0)\,B)$, where by induction we assume $\mathcal{F}(0)$ has already been defined. So $\mathcal{F}$ will preserve face maps on the nose simply by construction.

– $\mathcal{Y}(\mathsf{f}_l(k,\star)) \circ \varepsilon_l(k) = \mathsf{id}_{\mathcal{F}(l)}$
– $\mathcal{Y}(\mathsf{f}_{l+1}(i,\star)) \circ \varepsilon_{l+1}(j+1) = \varepsilon_l(j) \circ \mathcal{X}(\mathsf{f}(i,\star))$ *if* $i \leq j$
– $\mathcal{Y}(\mathsf{f}_{l+1}(i+1,\star)) \circ \varepsilon_{l+1}(j) = \varepsilon_l(j) \circ \mathcal{X}(\mathsf{f}_l(i,\star))$ *if* $j \leq i$

Figure 1: Coherence conditions for the exchange of face maps and degeneracies

– $\mathcal{Y}(\mathsf{f}_l(k,\star)) \circ \upsilon_l(k,\star) = \mathsf{id}_{\mathcal{F}(l)}$
– $\mathcal{Y}(\mathsf{f}_l(k+1,\star)) \circ \upsilon_l(k,\star) = \mathsf{id}_{\mathcal{F}(l)}$
– $\mathcal{Y}(\mathsf{f}_{l+1}(k,\overline{\star})) \circ \upsilon_{l+1}(k,\star) = \varepsilon_l(k) \circ \mathcal{X}(\mathsf{f}_l(k,\overline{\star}))$
– $\mathcal{Y}(\mathsf{f}_{l+1}(k+1,\overline{\star})) \circ \upsilon_{l+1}(k,\star) = \varepsilon_l(k) \circ \mathcal{X}(\mathsf{f}_l(k,\overline{\star}))$
– $\mathcal{Y}(\mathsf{f}_{l+1}(i,\star_1)) \circ \upsilon_{l+1}(j+1,\star_2) = \upsilon_l(j,\star_2) \circ \mathcal{X}(\mathsf{f}_l(i,\star_1))$ *if* $i \leq j$
– $\mathcal{Y}(\mathsf{f}_{l+1}(i+1,\star_1)) \circ \upsilon_{l+1}(j,\star_2) = \upsilon_l(j,\star_2) \circ \mathcal{X}(\mathsf{f}_l(i,\star_1))$ *if* $j < i$

Figure 2: Coherence conditions for the exchange of face maps and connections

– $\big(\varepsilon_{l+1}(i+1) \circ \mathcal{X}(\mathsf{d}_l(j))\big) \circ \big(\mathcal{Y}(\mathsf{d}_{l+1}(i+1)) \circ \varepsilon_l(j)\big) = \big(\varepsilon_{l+1}(j) \circ \mathcal{X}(\mathsf{d}_l(i))\big) \circ \big(\mathcal{Y}(\mathsf{d}_{l+1}(j)) \circ \varepsilon_l(i)\big)$ *if* $j \leq i$
– $\big(\upsilon_{l+1}(k,\star) \circ \mathcal{X}(\mathsf{d}_l(k))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(k,\star)) \circ \varepsilon_l(k)\big) = \big(\varepsilon_{l+1}(k) \circ \mathcal{X}(\mathsf{d}_l(k))\big) \circ \big(\mathcal{Y}(\mathsf{d}_{l+1}(k)) \circ \varepsilon_l(k)\big)$
– $\big(\upsilon_{l+1}(j,\star) \circ \mathcal{X}(\mathsf{d}_l(i))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(j,\star)) \circ \varepsilon_l(i)\big) = \big(\varepsilon_{l+1}(i+1) \circ \mathcal{X}(\mathsf{c}_l(j,\star))\big) \circ \big(\mathcal{Y}(\mathsf{d}_{l+1}(i+1)) \circ \upsilon_l(j,\star)\big)$ *if* $j < i$
– $\big(\upsilon_{l+1}(j+1,\star) \circ \mathcal{X}(\mathsf{d}_l(i))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(j+1,\star)) \circ \varepsilon_l(i)\big) = \big(\varepsilon_{l+1}(i) \circ \mathcal{X}(\mathsf{c}_l(j,\star))\big) \circ \big(\mathcal{Y}(\mathsf{d}_{l+1}(i)) \circ \upsilon_l(j,\star)\big)$ *if* $i \leq j$
– $\big(\upsilon_{l+1}(k,\star) \circ \mathcal{X}(\mathsf{c}_l(k,\star))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(k,\star)) \circ \upsilon_l(k,\star)\big) = \big(\varepsilon_{l+1}(k+1,\star) \circ \mathcal{X}(\mathsf{c}_l(k,\star))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(k+1,\star)) \circ \varepsilon_l(k,\star)\big)$
– $\big(\upsilon_{l+1}(i,\star_1) \circ \mathcal{X}(\mathsf{c}_l(j,\star_2))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(i,\star_1)) \circ \upsilon_l(j,\star_2)\big) = \big(\upsilon_{l+1}(j+1,\star_2) \circ \mathcal{X}(\mathsf{c}_l(i,\star_1))\big) \circ \big(\mathcal{Y}(\mathsf{c}_{l+1}(j+1,\star_2)) \circ \upsilon_l(i,\star_1)\big)$ *if* $i < j$

Figure 3: Coherence conditions for the exchange of degeneracies and/or connections

By contrast, strict equality for degeneracies and connections will not be attainable in most cases of interest. For instance, as we observe formally in Section 4, at each level $l = 0, 1$ the category $\mathsf{Rel}(l)$ has products $\times_l$. Hence, for two cubical functors $\mathcal{F}, \mathcal{G} : \mathsf{Rel} \to \mathsf{Rel}$ it makes sense to define the product $\mathcal{F} \times \mathcal{G}$ levelwise by $(\mathcal{F} \times \mathcal{G})(l) \coloneqq \mathcal{F}(l) \times_l \mathcal{G}(l)$. However, $\mathcal{F} \times \mathcal{G}$ will not preserve the degeneracy d on the nose even if $\mathcal{F}$ and $\mathcal{G}$ do since the identity type on a product is in general not *identical* to the product of identity types, only suitably *isomorphic*. The same is true for exponentials $\mathcal{F} \Rightarrow \mathcal{G}$.

Once we allow non-strict preservation of degeneracies and connections, the coherence conditions in Figures 1, 2, and 3 are natural counterparts to the cubical equalities in $\square_p$. For instance, the left-hand side of the first coherence condition in Figure 3 can be illustrated as the pasting of two rectangles:

$$
\begin{array}{ccc}
\mathcal{X}(l) & \xrightarrow{\quad \mathcal{F}(l) \quad} & \mathcal{Y}(l) \\
{\scriptstyle \mathcal{X}(\mathsf{d}_l(j))}\big\downarrow & \Downarrow \varepsilon_l(j) & \big\downarrow{\scriptstyle \mathcal{Y}(\mathsf{d}_l(j))} \\
\mathcal{X}(l+1) & \xrightarrow{\quad \mathcal{F}(l+1) \quad} & \mathcal{Y}(l+1) \\
{\scriptstyle \mathcal{X}(\mathsf{d}_{l+1}(i+1))}\big\downarrow & \Downarrow \varepsilon_{l+1}(i+1) & \big\downarrow{\scriptstyle \mathcal{Y}(\mathsf{d}_{l+1}(i+1))} \\
\mathcal{X}(l+2) & \xrightarrow[\quad \mathcal{F}(l+2) \quad]{} & \mathcal{Y}(l+2)
\end{array}
$$

Similarly, the right-hand-side of the first coherence condition in Figure 3 corresponds to the pasting of two more rectangles:

$$
\begin{array}{ccc}
\mathcal{X}(l) & \xrightarrow{\quad \mathcal{F}(l) \quad} & \mathcal{Y}(l) \\
{\scriptstyle \mathcal{X}(\mathsf{d}_l(i))}\big\downarrow & \Downarrow \varepsilon_l(i) & \big\downarrow{\scriptstyle \mathcal{Y}(\mathsf{d}_l(i))} \\
\mathcal{X}(l+1) & \xrightarrow{\quad \mathcal{F}(l+1) \quad} & \mathcal{Y}(l+1) \\
{\scriptstyle \mathcal{X}(\mathsf{d}_{l+1}(j))}\big\downarrow & \Downarrow \varepsilon_{l+1}(j) & \big\downarrow{\scriptstyle \mathcal{Y}(\mathsf{d}_{l+1}(j))} \\
\mathcal{X}(l+2) & \xrightarrow[\quad \mathcal{F}(l+2) \quad]{} & \mathcal{Y}(l+2)
\end{array}
$$

Since $j \leq i$, the cubical equality

$$\mathsf{d}_{l+1}(i+1) \circ \mathsf{d}_l(j) = \mathsf{d}_{l+1}(j) \circ \mathsf{d}_l(i)$$

guarantees that the respective sides of the two outer rectangles coincide. This leads to the natural requirement that the two pastings coincide.

Definition 6. *Let $(\mathcal{F}, \varepsilon, \upsilon)$ and $(\mathcal{G}, \epsilon, \nu)$ be p-dimensional cubical functors from $\mathcal{X}$ to $\mathcal{Y}$. A p-*dimensional cubical natural transformation *$\eta : \mathcal{F} \to \mathcal{G}$ is a family of natural transformations $\{\eta(l) : \mathcal{F}(l) \to G(l)\}_{l \leq p}$ that respects face maps, degeneracies, and connections in the following sense:*

- *For any $l, k \in \mathbb{N}$ with $k \leq l < p$ and $\star \in 2$, we have*

$$\mathcal{Y}(\mathsf{f}_l(k,\star)) \circ \eta(l+1) = \eta(l) \circ \mathcal{X}(\mathsf{f}_l(k,\star))$$

- *For any $l, k \in \mathbb{N}$ with $k \leq l < p$, the following diagram of natural transformations commutes:*

$$\begin{CD}
\mathcal{Y}(\mathsf{d}_l(k)) \circ \mathcal{F}(l) @>{\varepsilon_l(k)}>> \mathcal{F}(l+1) \circ \mathcal{X}(\mathsf{d}_l(k)) \\
@V{\mathcal{Y}(\mathsf{d}_l(k)) \circ \eta(l)}VV @VV{\eta(l+1) \circ \mathcal{X}(\mathsf{d}_l(k))}V \\
\mathcal{Y}(\mathsf{d}_l(k)) \circ \mathcal{G}(l) @>>{\epsilon_l(k)}> \mathcal{G}(l+1) \circ \mathcal{X}(\mathsf{d}_l(k))
\end{CD}$$

- *For any $l, k \in \mathbb{N}$ with $k \leq l < p$ and $\star \in 2$, the following diagram of natural transformations commutes:*

$$\begin{CD}
\mathcal{Y}(\mathsf{c}_l(k,\star)) \circ \mathcal{F}(l) @>{\upsilon_l(k,\star)}>> \mathcal{F}(l+1) \circ \mathcal{X}(\mathsf{c}_l(k,\star)) \\
@V{\mathcal{Y}(\mathsf{c}_l(k,\star)) \circ \eta(l)}VV @VV{\eta(l+1) \circ \mathcal{X}(\mathsf{c}_l(k,\star))}V \\
\mathcal{Y}(\mathsf{c}_l(k,\star)) \circ \mathcal{G}(l) @>>{\nu_l(k,\star)}> \mathcal{G}(l+1) \circ \mathcal{X}(\mathsf{c}_l(k,\star))
\end{CD}$$

Identity and composition for cubical natural transformations is defined levelwise. The identity cubical functor is defined in the obvious way; for composition we have:

Definition 7. *The composition $(\mathcal{G} \circ \mathcal{F}, \epsilon \circ \varepsilon, \nu \circ \upsilon)$ of cubical functors $(\mathcal{F}, \varepsilon, \upsilon) : \mathcal{X} \to \mathcal{Y}$ and $(\mathcal{G}, \epsilon, \nu) : \mathcal{Y} \to \mathcal{Z}$ is defined as follows:*

- $(\mathcal{G} \circ \mathcal{F})(l) \coloneqq \mathcal{G}(l) \circ \mathcal{F}(l)$
- $(\epsilon \circ \varepsilon)_l(k) \coloneqq (\mathcal{G}(l+1) \circ \varepsilon) \circ (\epsilon \circ \mathcal{F}(l))$
- $(\nu \circ \upsilon)_l(k,\star) \coloneqq (\mathcal{G}(l+1) \circ \upsilon) \circ (\nu \circ \mathcal{F}(l))$

The composition $\epsilon \circ \varepsilon$ can be seen as the pasting

$$\begin{CD}
\mathcal{X}(l) @>{\mathcal{F}(l)}>> \mathcal{Y}(l) @>{\mathcal{G}(l)}>> \mathcal{Z}(l) \\
@V{\mathcal{X}(\mathsf{d}_l(j))}VV \Downarrow \varepsilon_l(j) @V{\mathcal{Y}(\mathsf{d}_l(j))}VV \Downarrow \epsilon_l(j) @VV{\mathcal{Z}(\mathsf{d}_l(j))}V \\
\mathcal{X}(l+1) @>>{\mathcal{F}(l+1)}> \mathcal{Y}(l+1) @>>{\mathcal{G}(l+1)}> \mathcal{Z}(l+1)
\end{CD}$$

and similarly for $\nu \circ \upsilon$.

For cubical functors $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{Y} \to \mathcal{Z}$, and natural transformations $\eta_1 : \mathcal{F}_1 \to \mathcal{F}_2$ and $\eta_2 : \mathcal{G}_1 \to \mathcal{G}_2$, the compositions $\eta_1 \circ \mathcal{F}_1 : \mathcal{G}_1 \circ \mathcal{F}_1 \to \mathcal{G}_2 \circ \mathcal{F}_1$ and $\mathcal{G}_1 \circ \eta_2 : \mathcal{G}_1 \circ \mathcal{F}_1 \to \mathcal{G}_1 \circ \mathcal{F}_2$ are defined levelwise in the obvious way. For $n, i \in \mathbb{N}$ with $1 \leq i < n$, there is a strict cubical functor $\pi_i^n : \mathcal{X}^n \to \mathcal{X}$ corresponding to the i-th *projection*. Given cubical functors $\mathcal{F}_1, \ldots, \mathcal{F}_n : \mathcal{X} \to \mathcal{Y}$ we have the obvious cubical functor $\langle \mathcal{F}_1, \ldots, \mathcal{F}_n \rangle : \mathcal{X} \to \mathcal{Y}^n$ and given cubical natural transformations $\eta_1 : \mathcal{F}_1 \to \mathcal{G}_1, \ldots, \eta_n : \mathcal{F}_n \to \mathcal{G}_n$, we have the obvious cubical natural transformation $\langle \eta_1, \ldots, \eta_n \rangle : \langle \mathcal{F}_1, \ldots, \mathcal{F}_n \rangle \to \langle \mathcal{G}_1, \ldots, \mathcal{G}_n \rangle$.

4 Cartesian Closed Cubical Categories with Isomorphisms

We now impose some additional structure on cubical categories that will allow us to interpret the simply typed fragment of System F. We recall that a functor $F : C \to D$ between two cartesian closed categories is said to be *cartesian* if for any objects X, Y in C, the canonical morphisms $F(1) \to 1$, $F(X \times Y) \to (F(X) \times F(Y))$, $F(X \Rightarrow Y) \to (F(X) \Rightarrow F(Y))$ are invertible. In the case when they are identities, F will be called *strictly cartesian.* We denote the category of cartesian closed categories and cartesian functors internal to an ambient category $\mathcal{C}$ by $\mathsf{CCC}(\mathcal{C})$.

Definition 8. *A p-dimensional* cartesian closed *cubical category over $\mathcal{C}$ is a functor $\mathcal{X} : \square_p \to \mathsf{CCC}(\mathcal{C})$ where every face map $\mathcal{X}(\mathsf{f}_l(k,\star))$ is strictly cartesian.*

Example 9 (Reynolds' model, continued)**.** *The cartesian structure on the cubical category of sets and relations is as expected. The terminal relation relates every pair by the canonical term of* $\mathsf{1}$*. The product $R \times S$ of relations R and S relates a pair $\big((r_\top, s_\top), (r_\bot, s_\bot)\big)$ by a product (p, q) of witnesses $p : R(r_\top, r_\bot)$, $q : S(s_\top, s_\bot)$. The exponential $R \Rightarrow S$ of relations R and S relates a pair $(f_\top, f_\bot)$ of functions by a function of witnesses $q : \Pi_{(r_\top, r_\bot)} \Pi_{p:R(r_\top, r_\bot)} S\big(f_\top(r_\top), f_\bot(r_\bot)\big)$.*

Example 10 (Proof-relevant Reynolds' model, continued)**.** *The cartesian structure on the cubical category of sets, relations, and 2-relations is analogous. The terminal 2-relation relates every quadruple by the canonical term of* $\mathsf{1}$*. The product $R \times S$ of 2-relations R and S relates a quadruple $\big((r_{0\top}, s_{0\top}), (r_{1\top}, s_{1\top}), (r_{0\bot}, s_{0\bot}), (r_{1\bot}, s_{1\bot})\big)$ by a product (p, q) of witnesses, such that $p : R(r_{0\top}, r_{1\top}, r_{0\bot}, r_{1\bot})$ and $q : S(s_{0\top}, s_{1\top}, s_{0\bot}, s_{1\bot})$. The exponential $R \Rightarrow S$ of 2-relations R and S relates a quadruple $(f_{0\top}, f_{1\top}, f_{0\bot}, f_{1\bot})$ of functions by a function of witnesses $q : \Pi_{(r_{0\top}, r_{1\top}, r_{0\bot}, r_{1\bot})} \Pi_{p:R(r_{0\top}, r_{1\top}, r_{0\bot}, r_{1\bot})} \ S\big(f_{0\top}(r_{0\top}), f_{1\top}(r_{1\top}), f_{0\bot}(r_{0\bot}), f_{1\bot}(r_{1\bot})\big)$.*

As we observed earlier, degeneracies and connections in our two running examples do not preserve the cartesian structure on the nose, only up to suitable isomorphisms. As explained in [29], this means we can no longer interpret a type $\alpha \vdash T(\alpha)$ as a *discrete* functor $[\![T]\!] : |\mathsf{Rel}| \to \mathsf{Rel}$ as is done, *e.g.*, in [12]. Instead, we need to ensure that $[\![T]\!]$ preserves enough isomorphisms, including the ones witnessing the preservation of the cartesian structure. To this end we introduce:

Definition 11. *Given a p-dimensional cubical category $\mathcal{X}$ over an ambient category $\mathcal{C}$, a* cubical subcategory *of $\mathcal{X}$ is another p-dimensional cubical category $\mathcal{M}$ over $\mathcal{C}$, together with a strict cubical functor $\mathcal{I} : \mathcal{M} \to \mathcal{X}$ such that for each $l \leq p$, the arrows $\mathcal{I}(l)_0$ and $\mathcal{I}(l)_1$ of $\mathcal{C}$ are monomorphisms. If each $\mathcal{I}(l)_0$ is in fact an isomorphism, we call this subcategory* wide*.*

The category $\mathcal{M}(l)$ serves to select the *relevant morphisms* of $\mathcal{X}(l)$, *i.e.*, the ones whose preservation we care about. We consider a morphism of $\mathcal{X}(l)$ relevant if it lies in the image of $\mathcal{I}(l)_1$. Clearly, all identity morphisms are relevant and the composition of two relevant morphisms is again relevant.

We now come to the main definition of this section, which encapsulates all the information needed to interpret the simply typed fragment of System F in our setting:

Definition 12. *A p-dimensional* cartesian closed cubical category with isomorphisms *is a cartesian closed cubical category $\mathcal{X}$ with a wide cubical subcategory $\mathcal{M} \overset{\mathcal{I}}{\hookrightarrow} \mathcal{X}$ such that:*

- *Every morphism in $\mathcal{M}(l)$ for $l \leq p$ is invertible.*
- *The relevant morphisms in $\mathcal{X}(l)$ for $l \leq p$ are closed under products and exponentiation in $\mathcal{X}(l)$,* i.e.*, if f and g are relevant, then so are $f \times g$ and $f \Rightarrow g$.*
- *The canonical isomorphisms witnessing the preservation of the cartesian structure by each degeneracy $\mathsf{d}_l(k)$ and each connection $\mathsf{c}_l(k,\star)$ are all relevant.*

In particular, in the setting of Definition 12, any relevant morphism is an isomorphism (hence the choice of terminology) and its inverse is also relevant. One obvious way to turn a cubical category $\mathcal{X}$ into a cubical category with isomorphisms is to take $\mathcal{M} := |\mathcal{X}|$, so that the only relevant isomorphisms are the identity morphisms. The resulting cubical category with isomorphisms is then cartesian closed if and only if the degeneracies and connections preserve the existing cartesian structure on $\mathcal{X}$ *on the nose*. Thus, the strict case (suitable for *e.g.*, the PER model of Longo and Moggi [19]) is a special case of our definition. In our running examples we go to the other extreme and take *all* isomorphisms to be relevant:

Example 13 (Both Reynolds' models, continued)**.** *We construct a cubical subcategory* $\mathsf{Iso} \hookrightarrow \mathsf{Rel}$ *as follows. The objects of $\mathsf{Iso}(l)$ are the objects of $\mathsf{Rel}(l)$,* i.e.*, $\mathsf{Iso}(l)_0 := \mathsf{Rel}(l)_0$, and the morphisms of $\mathsf{Iso}(l)$ are the isomorphisms of $\mathsf{Rel}(l)$,* i.e.*, $\mathsf{Iso}(l)_1$ is the type*

$$\big\{(i, j) : \mathsf{Rel}(l)_1 \times \mathsf{Rel}(l)_1 \ \& \mathsf{s}(i) = \mathsf{t}(j) \ \times \ \mathsf{t}(i) = \mathsf{s}(j) \ \times \ j \circ i = \mathsf{id} \ \times \ i \circ j = \mathsf{id}\big\}$$

where $\mathsf{s}, \mathsf{t}, \mathsf{id}, \circ$ refer to the source, target, identity, and composition in $\mathsf{Rel}(l)$, and we write $\{x : A \ \& \ B(x)\}$ for $\Sigma_{x:A} B(x)$ to enhance readability. The first (or second) projection gives the required mono from $\mathsf{Iso}(l)_1$ to $\mathsf{Rel}(l)_1$.

5 Cubical Models of Parametricity

As we mentioned previously, cartesian closed cubical categories with isomorphisms provide all the ingredients to interpret the simply typed fragment of System F. To show how, we recall some notions from [29]:

Definition 14. *A* wide subfibration *of a split fibration $U : \mathcal{E} \to \mathcal{B}$ is a restriction $U' : \mathcal{E}' \to \mathcal{B}$ of U, where $\mathcal{E}'$ is a wide subcategory of $\mathcal{E}$ with the property that, for any object X of $\mathcal{E}$ and $f : Y \to UX$ in $\mathcal{B}$, the cartesian lifting of f with respect to U is cartesian with respect to U'.*

A wide subfibration is the fibrational analogue of a wide cubical subcategory and likewise serves to select the relevant isomorphisms. To give the fibrational analogue of a cartesian closed cubical category with isomorphisms, the following notation will be useful:

Notation 15. *Let $\mathcal{B}$ be a category whose objects are in a bijection with the natural numbers, with $n+1$ serving as a product of n and 1.*

- *For $n, k \in \mathbb{N}$, denote by $\mathsf{p}_n(k) : n+k+1 \to n+k$ the "weakening morphism" that drops the k^{th} variable in the context, counting from the right, defined inductively by*

$$\mathsf{p}_n(k) \coloneqq \pi_1$$
$$\mathsf{p}_n(k+1) \coloneqq \mathsf{p}_n(k) \times \mathsf{id}$$

- *For $n, k \in \mathbb{N}$ and a morphism $A : n \to 1$, denote by $\mathsf{s}_n(k, A) : n+k \to n+k+1$ the "substitution morphism" that substitutes A for the k^{th} variable, defined inductively by*

$$\mathsf{s}_n(k, A) \coloneqq \langle \mathsf{id}, A \rangle$$
$$\mathsf{s}_n(k+1, A) \coloneqq \mathsf{s}_n(k, A) \times \mathsf{id}$$

Definition 16. *A* fibration with isomorphisms *is a split fibration $U : \mathcal{E} \to \mathcal{B}$ (the* underlying fibration*), together with a wide subfibration $U' : \mathcal{E}' \to \mathcal{B}$ of U (the* fibration of isomorphisms*), satisfying the following properties:*

1. *The objects of $\mathcal{B}$ are in bijection with the natural numbers, with 0 serving as a terminal object in $\mathcal{B}$, 1 serving as a split generic object for U, and $n+1$ serving as a product of n and 1.*
2. *For $n \in \mathbb{N}$, every morphism in $\mathcal{E}'_n$ is an isomorphism.*
3. *For $n, k \in \mathbb{N}$ and morphism $i : A \to B$ in $\mathcal{E}'_n$, there is a natural transformation $\phi_n(k, i)$ between the substitution functors $\mathsf{s}(k, A)^*$ and $\mathsf{s}_n(k, B)^* : \mathcal{E}_{n+k+1} \to \mathcal{E}_{n+k}$ with components in $\mathcal{E}'$ such that:*
   (a) $\phi_n(0, i)\, \mathsf{q}_n = i$
   (b) $\phi_n(k+1, i)\, \mathsf{q}_{n+k+1} = \mathsf{id}_{\mathsf{q}_{n+k}}$
   (c) $\phi_n(0, i)\, (\mathsf{p}_n(0)^*\, X) = \mathsf{id}_X$ *for every object X in $\mathcal{E}_n$*
   (d) $\phi_n(k+1, i)\, (\mathsf{p}_{n+k+1}(0)^*\, X) = \mathsf{p}_{n+k}(0)^*\, (\phi_n(k, i)\, X)$ *for every object X in $\mathcal{E}_{n+k+1}$*

Condition 2 justifies our choice of terminology in Definition 16. Conditions 3a through 3d ensure that when X is the interpretation of a System F type, $\phi_n(k, i)\, X$ is precisely the isomorphism determined by induction on the structure of X. Similar conditions are needed any time we impose more structure on the fibration. For example:

Definition 17. *A* cartesian closed fibration with isomorphisms *is a fibration with isomorphisms such that:*

1. *For $n \in \mathbb{N}$, the fiber $\mathcal{E}_n$ is cartesian closed, with a terminal object 1_n, products $\times_n$, and exponentials $\Rightarrow_n$, and products and exponentials preserve membership in $\mathcal{E}'$.*
2. *Beck-Chevalley: for a morphism $f : n \to m$ in $\mathcal{B}$ and objects X, Y in $\mathcal{E}_m$, the canonical morphisms below are in $\mathcal{E}'$:*

$$\theta_1(f) : f^*(1_m) \to 1_n$$
$$\theta_\times(f, X, Y) : f^*(X \times_m Y) \to \big(f^*(X) \times_n f^*(Y)\big)$$
$$\theta_\Rightarrow(f, X, Y) : f^*(X \Rightarrow_m Y) \to \big(f^*(X) \Rightarrow_n f^*(Y)\big)$$

3. *For $n, k \in \mathbb{N}$, morphism $i : A \to B$ in $\mathcal{E}'_n$, and objects X, Y in $\mathcal{E}_{n+k+1}$, we have*

$$\theta_1(\mathsf{s}_n(k, B)) \circ \phi_n(k, i)(1_{n+k+1}) = \theta_1(\mathsf{s}_n(k, A))$$

$$\theta_\times\big(\mathsf{s}_n(k, B), X, Y\big) \circ \phi_n(k, i)\,(X \times_{n+k+1} Y) = \big((\phi_n(k, i)\,X) \times_{n+k} (\phi_n(k, i)\,Y)\big) \circ \theta_\times\big(\mathsf{s}_n(k, A), X, Y\big)$$

$$\theta_\Rightarrow\big(\mathsf{s}_n(k, B), X, Y\big) \circ \phi_n(k, i)\,(X \Rightarrow_{n+k+1} Y) = \big((\phi_n(k, i)\,X)^{-1} \Rightarrow_{n+k} (\phi_n(k, i)\,Y)\big) \circ \theta_\Rightarrow\big(\mathsf{s}_n(k, A), X, Y\big)$$

Our main technical result shows that any cartesian closed cubical category $\mathcal{R}$ canonically interprets the simply typed fragment of System F; see [17] for the proof.

Theorem 18. *Given a p-dimensional cartesian closed cubical category with isomorphisms $\mathcal{R} \coloneqq \mathcal{M} \overset{\mathcal{I}}{\hookrightarrow} \mathcal{X}$, there is a canonical cartesian closed fibration with isomorphisms $\int \mathcal{R}$, where substitution preserves the cartesian closed structure on the nose,* i.e., *this fibration is split cartesian.*

The objects in the fiber over n are p-dimensional cubical functors $\mathcal{M}^n \to \mathcal{M}$ and morphisms from $\mathcal{F}$ to $\mathcal{G}$ in the fiber over n are p-dimensional cubical natural transformations from $\mathcal{I} \circ \mathcal{F}$ to $\mathcal{I} \circ \mathcal{G}$.

Since any split cartesian closed fibration induces a canonical interpretation of the simply typed fragment of System F, the above theorem leads to an interpretation of a type $\Delta \vdash T$ as a cubical functor $[\![T]\!] : \mathcal{M}^{|\Delta|} \to \mathcal{M}$; in particular, $[\![T]\!](l)$ acts functorially on all relevant isomorphisms. The interpretation of a term $\Delta; S \vdash t : T$ is as a cubical natural transformation $[\![t]\!]$ from $\mathcal{I} \circ [\![S]\!]$ to $\mathcal{I} \circ [\![T]\!]$, *i.e.*, the components of $[\![t]\!](l)$ are themselves morphisms in $\mathcal{X}(l)$, just as originally intended by Reynolds, while naturality is with respect to all relevant isomorphisms.

To interpret full System F, we invoke the following from [29]:

Definition 19. *A* λ2-fibration with isomorphisms *is a cartesian closed fibration with isomorphisms such that:*

1. *For $n \in \mathbb{N}$, the weakening functor $\mathsf{p}_n(0)^* : \mathcal{E}_n \to \mathcal{E}_{n+1}$ has a right adjoint $\forall_n$, and these adjoints preserve membership in $\mathcal{E}'$.*
2. *Beck-Chevalley: for a morphism $f : n \to m$ in $\mathcal{B}$ and object X in $\mathcal{E}_{m+1}$, the canonical morphism below is in $\mathcal{E}'$:*

$$\theta_\forall(f, X) : f^*(\forall_m(X)) \to \forall_n((f \times \mathsf{id})^*(X))$$

3. *For $n, k \in \mathbb{N}$, morphism $i : A \to B$ in $\mathcal{E}'_n$, and object X in $\mathcal{E}_{n+k+2}$, we have*

$$\theta_\forall\big(\mathsf{s}_n(k, B), X\big) \circ \phi_n(k, i)\,(\forall_{n+k+1}(X)) = \forall_{n+k}(\phi_n(k+1, i)\,X) \circ \theta_\forall\big(\mathsf{s}_n(k, A), X\big)$$

Theorem 20. *Every λ2-fibration $U : \mathcal{E} \to \mathcal{B}$ with isomorphisms gives a sound model of System F in which:*

- *every type context Γ is interpreted as an object $[\![\Gamma]\!]$ in $\mathcal{B}$*
- *every type $\Gamma \vdash T$ is interpreted as an object $[\![\Gamma \vdash T]\!]$ in the fiber $\mathcal{E}_{[\![\Gamma]\!]}$*
- *every term context $\Gamma; \Delta$ is interpreted as an object $[\![\Gamma \vdash \Delta]\!]$ in the fiber $\mathcal{E}_{[\![\Gamma]\!]}$*
- *every term $\Gamma; \Delta \vdash t : T$ is interpreted as a morphism $[\![\Gamma; \Delta \vdash t : T]\!]$ from $[\![\Gamma; \Delta]\!]$ to $[\![\Gamma \vdash T]\!]$ in the fiber $\mathcal{E}_{[\![\Gamma]\!]}$*

We can now formulate our main definition, which is a p-dimensional generalization of the corresponding notion from [29]:

Definition 21. *Let $p \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R}$ be a p-dimensional cartesian closed cubical category with isomorphisms. A model of System F given as a λ2-fibration with isomorphisms U is* p-parametric *over $\mathcal{R}$ if there is a functor $\mu : U \to \int \mathcal{R}$ of cartesian closed fibrations with isomorphisms whose restriction to the fibers of U and $\int \mathcal{R}$ over the terminal objects is full, faithful, and essentially surjective.*

The λ2-fibration U can be seen as a representation of the cubical functors and natural transformations in $\int \mathcal{R}$ that allows us to formulate the adjunctions needed to interpret $\forall$-types. The motivating example for the above definition is the term model of System F, where the fiber of U over $n \in \mathbb{N}$ consists of types and terms with n free type variables. The category $\mathcal{R}$ of closed System F types and terms can be seen as a 0-dimensional cartesian closed cubical category with (trivial) isomorphisms over $\mathcal{C} \coloneqq \mathsf{Set}$. The fiber of $\int \mathcal{R}$ over n thus consists of discrete functors

$|\mathcal{R}|^n \to \mathcal{U}$ and (vacuously) natural transformations between them. Since these functions are set-theoretic, $\int \mathcal{R}$ does not admit the adjunctions needed to interpret $\forall$-types. The term model *does*, however, since there we have the extra information that the functions in fact come from syntax, as types and terms with free variables.

In an *empty* context there is no difference between an *ad hoc* function and a polymorphic one. Thus, we expect the respective fibers of U and $\int \mathcal{R}$ over the terminal objects to be equivalent, as is indeed the case, accounting for the fullness, faithfulness, and essential surjectivity requirement of Definition 21.

Corollary 22. *The term model of System F is* 0*-parametric over the category of closed System F types and terms.*

Theorem 23. *Every p-parametric model of System F over a cartesian closed cubical category with isomorphisms* $\mathcal{M} \overset{\mathcal{I}}{\hookrightarrow} \mathcal{X}$ *is a sound model in which:*

- *any type* $\Delta \vdash T$ *induces a p-dimensional cubical functor* $[\![\Delta \vdash T]\!] : \mathcal{M}^{|\Delta|} \to \mathcal{M}$
- *any term* $\Delta; \Gamma \vdash t : T$ *induces a p-dimensional cubical natural transformation* $[\![\Delta; \Gamma \vdash t : T]\!] : \mathcal{I} \circ [\![\Delta \vdash \Gamma]\!] \to \mathcal{I} \circ [\![\Delta \vdash T]\!]$

Clearly, a p-dimensional cartesian closed cubical category with isomorphisms $\mathcal{R}$ can be restricted to just the levels $0, \ldots, l$ for $l < p$ to obtain an l-dimensional cartesian closed cubical category with isomorphisms $\mathcal{R}|_l$. Then we have:

Theorem 24. *Let* $l < p \in \mathbb{N} \cup \{\infty\}$*. Any model of System F that is p-parametric over* $\mathcal{R}$ *is l-parametric over* $\mathcal{R}|_l$*.*

Proof. It suffices to show that the obvious restriction functor from $\int \mathcal{R}$ to $\int \mathcal{R}|_l$ is full, faithful, and essentially surjective on the fibers over the terminal objects. The key observation for all three properties is the following: given a cartesian closed cubical category with isomorphisms $\mathcal{M} \overset{\mathcal{I}}{\hookrightarrow} \mathcal{X}$, cubical functors $(\mathcal{F}, \varepsilon, \upsilon)$, $(\mathcal{G}, \epsilon, \nu) : \mathcal{M}^0 \to \mathcal{M}$ and a natural transformation $\eta(0) : \mathcal{I}(0) \circ \mathcal{F}(0) \to \mathcal{I}(0) \circ \mathcal{G}(0)$, there is a unique way to extend $\eta(0)$ to a cubical natural transformation $\eta : \mathcal{I} \circ \mathcal{F} \to \mathcal{I} \circ \mathcal{G}$ by inductively defining

$$\eta(k+1) := \big(\mathcal{I}(k+1) \circ \epsilon_k(0)\big) \circ \big(\mathcal{X}(\mathsf{d}_k(0)) \circ \eta(k)\big) \circ \big(\mathcal{I}(k+1) \circ \varepsilon_k(0)\big)^{-1}$$

The cubical equalities and the corresponding coherence conditions ensure that we could have used any other degeneracy or connection instead of $\mathsf{d}_k(0)$. For instance: using $\mathsf{d}_{k+1}(i+1)$ instead of $\mathsf{d}_{k+1}(0)$ results in the following:

$$\begin{aligned}
\eta(k+2) &:= \big(\mathcal{I}(k+2) \circ \epsilon_{k+1}(i+1)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(i+1)) \circ \eta(k+1)\big) \circ \big(\mathcal{I}(k+2) \circ \varepsilon_{k+1}(i+1)\big)^{-1} \\
&\overset{(1)}{=} \big(\mathcal{I}(k+2) \circ \epsilon_{k+1}(i+1)\big) \circ \big(\mathcal{X}(\mathsf{d}_k(i+1)) \circ \mathcal{I}(k+1) \circ \epsilon_k(0)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(i+1)) \circ \mathcal{X}(\mathsf{d}_k(0)) \circ \eta(k)\big) \circ \\
&\quad \big(\mathcal{X}(\mathsf{d}_k(i+1)) \circ \mathcal{I}(k+1) \circ \varepsilon_k(0)\big)^{-1} \circ \big(\mathcal{I}(k+2) \circ \varepsilon_{k+1}(i+1)\big)^{-1} \\
&\overset{(2)}{=} \big(\mathcal{I}(k+2) \circ \epsilon_{k+1}(i+1)\big) \circ \big(\mathcal{I}(k+2) \circ \mathcal{M}(\mathsf{d}_k(i+1)) \circ \epsilon_k(0)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(i+1)) \circ \mathcal{X}(\mathsf{d}_k(0)) \circ \eta(k)\big) \circ \\
&\quad \big(\mathcal{I}(k+2) \circ \mathcal{M}(\mathsf{d}_k(i+1)) \circ \varepsilon_k(0)\big)^{-1} \circ \big(\mathcal{I}(k+2) \circ \varepsilon_{k+1}(i+1)\big)^{-1} \\
&\overset{(3)}{=} \big(\mathcal{I}(k+2) \circ \epsilon_{k+1}(0)\big) \circ \big(\mathcal{I}(k+2) \circ \mathcal{M}(\mathsf{d}_{k+1}(0)) \circ \epsilon_k(i)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(0)) \circ \mathcal{X}(\mathsf{d}_k(i)) \circ \eta(k)\big) \circ \\
&\quad \big(\mathcal{I}(k+2) \circ \mathcal{M}(\mathsf{d}_{k+1}(0)) \circ \varepsilon_k(i)\big)^{-1} \circ \big(\mathcal{I}(k+2) \circ \varepsilon_{k+1}(0)\big)^{-1} \\
&\overset{(4)}{=} \big(\mathcal{I}(k+2) \circ \epsilon_{k+1}(0)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(0)) \circ \mathcal{I}(k+1) \circ \epsilon_k(i)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(0)) \circ \mathcal{X}(\mathsf{d}_k(i)) \circ \eta(k)\big) \circ \\
&\quad \big(\mathcal{X}(\mathsf{d}_{k+1}(0)) \circ \mathcal{I}(k+1) \circ \varepsilon_k(i)\big)^{-1} \circ \big(\mathcal{I}(k+2) \circ \varepsilon_{k+1}(0)\big)^{-1} \\
&\overset{(5)}{=} \big(\mathcal{I}(k+2) \circ \epsilon_{k+1}(0)\big) \circ \big(\mathcal{X}(\mathsf{d}_{k+1}(0)) \circ \eta(k+1)\big) \circ \big(\mathcal{I}(k+2) \circ \varepsilon_{k+1}(0)\big)^{-1}
\end{aligned}$$

The first equality follows by the inductive definition of $\eta(k+1)$, the second and fourth by definition of $\mathcal{I} : \mathcal{M} \to \mathcal{X}$ as a strict cubical functor, the third by the appropriate cubical equality and the corresponding coherence conditions on ϵ and ε, and the last by the inductive hypothesis.

Analogously, up to natural isomorphism there is a unique way to extend an arrow $\mathcal{H}(0) : \mathcal{M}(0)^0 \to \mathcal{M}(0)$ to a cubical functor $\mathcal{H} : \mathcal{M}^0 \to \mathcal{M}$ by inductively defining $\mathcal{H}(k+1) := \mathcal{M}(\mathsf{d}_k(0)) \circ \mathcal{H}(k)$, where, again, we could have used any other degeneracy or connection. ☐

6 Examples: Constructing Cubical Categories

In this section we use eCIC to construct a specific p-dimensional cubical category for *every* $p \in \mathbb{N}$. Our two running examples will arise as the cases $p = 1$ and $p = 2$.

At each level, the sets, relations, and 2-relations form a category, which we can express in eCIC as a tuple $(\mathsf{Ob}, \mathsf{Mor}, \mathsf{s}, \mathsf{t}, \mathsf{id}, \circ)$ with $\mathsf{Ob}, \mathsf{Mor} : \mathbb{U}_1$ the types of objects and morphisms, respectively. We can likewise formulate the notions of functors, cartesian closed categories, and (strict) cartesian functors in eCIC, all as expected. The category of cartesian closed eCIC categories and cartesian functors, which we denote by CCC, is isomorphic to the category $\mathsf{CCC}(\mathcal{C})$ in an obvious way, and this isomorphism preserves the strictness of cartesian functors. Therefore, any functor $\mathcal{X} : \square_p \to \mathsf{CCC}$ such that each $\mathcal{X}(\mathsf{f}_l(k, \star))$ is strictly cartesian defines a p-dimensional cartesian closed cubical category.

To see how to generalize our examples to higher dimensions, we recall that a relation at level 1 or 2 always consists of a *carrier* together with a predicate, either proof-relevant or proof-irrelevant, on the *elements* of the carrier. At level 1, the carrier of a relation R is a pair $(R_\top, R_\bot)$ of sets and the elements of this carrier are pairs $(r_\top, r_\bot) : R_\top \times R_\bot$. At level 2, the carrier of a relation Q is a quadruple $(Q_{0\top}, Q_{1\top}, Q_{0\bot}, Q_{1\bot})$ of 1-relations forming a square and the elements of this carrier are quadruples $(q_{0\top}, q_{1\top}, q_{0\bot}, q_{1\bot})$ likewise appropriately forming a square. This suggests the following abstraction:

Definition 25. *An* opfibration $\mathbf{A}$ *is a category in eCIC endowed with a function* $\widehat{(-)}$ *that associates to each object* X *a type* $\widehat{X} : \mathbb{U}$ *of* elements*, for instance the pairs* $(r_\top, r_\bot)$ *or quadruples* $(q^{0\top}, q^{1\top}, q^{0\bot}, q^{1\bot})$ *from above. Moreover,* $\widehat{(-)}$ *associates to each morphism* $f : X \to Y$ *a map* $\widehat{f} : \widehat{X} \to \widehat{Y}$*, in a manner consistent with identity and composition.*

An opfibration in eCIC is a special case of an opfibration in category theory for which each fiber $\widehat{X}$ is discrete.

Definition 26. *An opfibration* functor $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *is a functor between the underlying categories of* $\mathbf{A}$ *and* $\mathbf{B}$*, endowed with a natural family of maps* $\{\mathbf{F}_X : \widehat{X} \to \widehat{\mathbf{F}(X)}\}_X$ *indexed by objects* X *of* $\mathbf{A}$*. Naturality means that for any morphism* $f : X \to Y$ *in* $\mathbf{A}$ *the following diagram of functions commutes:*

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\ \widehat{f}\ } & \widehat{Y} \\ {\scriptstyle \mathbf{F}_X}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{F}_Y} \\ \widehat{\mathbf{F}(X)} & \xrightarrow[\ \widehat{\mathbf{F}(f)}\ ]{} & \widehat{\mathbf{F}(Y)} \end{array}$$

The identity and composition of functors is defined in the obvious way, with $(\mathbf{G} \circ \mathbf{F})_X \coloneqq \mathbf{G}_{\mathbf{F}(X)} \circ \mathbf{F}_X$*.*

We will sometimes omit the subscript X in expressions of the form $\mathbf{F}_X(x)$ if the object X is clear from the context.

Definition 27. *A* cartesian closed opfibration $\mathbf{A}$ *has a cartesian closed category as its underlying category. Moreover, for any object* X*, morphisms* $x : 1 \to X$ *can be identified with elements* $\widehat{x} : \widehat{X}$*, and this family of bijections is natural in the sense that for any* $x : 1 \to X$ *and* $M : X \to Y$*, we have* $\widehat{M \circ x} = \widehat{M}\ \widehat{x}$*.*

For the correspondence in the opposite direction, we write $\vec{x} : 1 \to X$ whenever $x : \widehat{X}$. The identification of elements of X with morphisms from the terminal object into X implies that the terminal object has a unique element. The naturality condition, analogous to the naturality condition for a split generic object in a split fibration (see Section 6), implies that $\widehat{x} : X$ is obtained by applying the function $\widehat{x} : \widehat{1} \to \widehat{X}$ to the unique element of 1, explaining the notation. The opfibrations of sets, relations, and 2-relations are clearly cartesian closed.

If A and B are objects in a cartesian closed opfibration, then any element $c : \widehat{A \times B}$ induces an element $c..1 : \widehat{A}$ by carrying c along the first projection, *i.e.*, $c..1 \coloneqq \widehat{\mathsf{fst}[A, B]}\, c$. We define $c..2 : \widehat{B}$ similarly. In the opposite direction, elements $a : \widehat{A}$ and $b : \widehat{B}$ induce an element $\langle a, b\rangle : \widehat{A \times B}$ as follows: since elements of an object X can be identified with morphisms $1 \to X$, we just need to show that morphisms $a : 1 \to A$ and $b : 1 \to B$ induce a morphism $\langle a, b\rangle : 1 \to A \times B$. Such a morphism is of course given by the universal property of the product. Finally, if A and B are objects in a cartesian closed opfibration, elements $f : \widehat{A \Rightarrow B}$ and $a : \widehat{A}$ induce an element $f[a] : \widehat{B}$ by carrying the pairing $\langle f, a\rangle$ along the evaluation morphism, *i.e.*, $f[a] \coloneqq \widehat{\mathsf{eval}[A, B]}\ \langle f, a\rangle$.

Definition 28. *An opfibration functor* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *between cartesian closed opfibrations is* (strictly) cartesian *if the underlying functor between the underlying cartesian closed categories of* $\mathbf{A}$ *and* $\mathbf{B}$ *is (strictly) cartesian, and for any element* $x : \widehat{X}$ *in* $\mathbf{A}$*, the following diagram in* $\mathbf{B}$ *commutes:*

$$\begin{array}{ccc} 1 & \xrightarrow{\ \overrightarrow{\mathbf{F}(x)}\ } & \\ {\scriptstyle \cong}\big\uparrow & & \mathbf{F}(X) \\ \mathbf{F}(1) & \xrightarrow[\ \mathbf{F}(\vec{x})\ ]{} & \end{array}$$

Here $\cong$ is the canonical isomorphism witnessing the preservation of the terminal object by $\mathbf{F}$. *In the strict case,* $\cong$ *is the identity so the above diagram reduces to the equality* $\mathbf{F}(\vec{x}) = \overrightarrow{\mathbf{F}(x)}$.

We denote the category of cartesian closed opfibrations and cartesian opfibration functors by CCOF. Since the obvious forgetful functor from CCOF to CCC preserves the strictness of cartesian functors, any functor $\mathcal{X} : \square_p \to \mathsf{CCOF}$ such that each $\mathcal{X}(\mathsf{f}_l(k,\star))$ is strictly cartesian defines a p-dimensional cartesian closed cubical category. We devote the rest of this section to constructing precisely such a functor $\mathsf{Rel} : \square_p \to \mathsf{CCOF}$ for any $p \in \mathbb{N}$.

We now introduce some important opfibrations which will be used in our construction. The simplest case of an opfibration is the terminal one:

Definition 29. *The* terminal opfibration $\mathbf{1}$ *has the unit type* $\mathsf{1}$ *as both the type of objects and morphisms. The type of elements assigned to the unique object is again* $\mathsf{1}$.

As suggested by our running examples, given a cartesian closed opfibration $\mathbf{A}$, we can construct a new cartesian closed opfibration by endowing each object X of $\mathbf{A}$ with a predicate on $\widehat{X}$. A morphism in this new opfibration is a morphism of $\mathbf{A}$ endowed with a map between the associated predicates. At levels $l < p$, the predicates are proof-relevant, whereas at the highest level $l = p$, if $p > 0$, they need to be proof-irrelevant. To uniformly cover both cases, we parametrize the construction over a *subuniverse* of $\mathbb{U}$, defined as a family $\mathcal{U} : \mathbb{U} \to \mathsf{Prop}$ of propositions that:

- is closed under products and terminal objects, in the sense that the type $\mathsf{1}$ is in $\mathcal{U}$ and so are the types $A \times B$ and $A \to B$ whenever A and B are in $\mathcal{U}$,
- is impredicative, in the sense that the type $\Pi_{a:A} B(x)$ is in $\mathcal{U}$ whenever for any $x : A : \mathbb{U}$ the type $B(x)$ is in $\mathcal{U}$ (this in particular implies closure under $\to$),
- subsumes identity types, in the sense that for any $A : \mathbb{U}$, the type $M =_A N$ is in $\mathcal{U}$.

When saying a type T is in $\mathcal{U}$ we mean the proposition $\mathcal{U}(T)$ is inhabited; the impredicativity in the second item above refers to the fact that A itself does not have to be in $\mathcal{U}$. The subuniverse $\mathcal{U}$ can itself be seen as a type $\Sigma_{T:\mathbb{U}} \mathcal{U}(T)$ and any term T in $\mathcal{U}$ can be seen as a type in $\mathbb{U}$.

There are two subuniverses of interest to us: $\mathcal{U}_{\mathsf{all}}$, which corresponds to $\mathbb{U}$, is defined by $\mathcal{U}_{\mathsf{all}}(T) \coloneqq \mathsf{1}$, with the type $\mathsf{1}$ seen as a proposition, and $\mathcal{U}_{\mathsf{prop}}$, which corresponds to Prop, is defined by $\mathcal{U}_{\mathsf{prop}}(T) \coloneqq \mathsf{isProp}(T)$, with $\mathsf{isProp}(T)$ again seen as a proposition.

Definition 30. *Let* $\mathbf{A}$ *be a cartesian closed opfibration with types* Ob *and* Mor *of objects and morphisms, respectively, and* $\mathcal{U}$ *be a subuniverse of* $\mathbb{U}$. *The* predicate opfibration $\mathbb{P}(\mathbf{A},\mathcal{U})$ *is a cartesian closed opfibration where objects are pairs* (X, P) *with* $X : \mathsf{Ob}$ *and* $P : \widehat{X} \to \mathcal{U}$, *and elements of* (X, P) *are pairs* (x, p) *with* $x : \widehat{X}$ *and* $p : P(x)$. *We call* X, x *the* carriers, *and* P, p *the* predicate parts *of* $R \coloneqq (X, P)$ *and* $r \coloneqq (x, p)$, *respectively, denoted* $R_{\mathbb{C}}, r_{\mathbb{C}}$ *and* $R_{\mathbb{P}}, r_{\mathbb{P}}$.

The morphisms from R *to* S *are pairs* (f, g), *with the* carrier $f : \mathsf{Mor}$ *a morphism from* $R_{\mathbb{C}}$ *to* $S_{\mathbb{C}}$, *and* $g : \widehat{R} \to \widehat{S}$ *maps elements in a manner consistent with* f, *i.e.,* $g(r)_{\mathbb{C}} = f(r_{\mathbb{C}})$. *We let* $\widehat{(f, g)} \coloneqq g$.

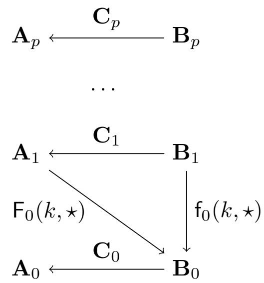
The terminal object in $\mathbb{P}(\mathbf{A},\mathcal{U})$ *pairs up the terminal object* $\mathsf{1}$ *in* $\mathbf{A}$ *with the predicate* $x \mapsto \mathsf{1}$ *where the unit type is seen as a type in* $\mathcal{U}$. *The product of* R *and* S *pairs up the product of carriers* $R_{\mathbb{C}} \times S_{\mathbb{C}}$ *with the predicate* $z \mapsto R_{\mathbb{P}}(z..1) \times S_{\mathbb{P}}(z..2)$, *where the product is again seen as a type in* $\mathcal{U}$. *Finally, the exponential of* R *and* S *pairs up the exponential of carriers* $R_{\mathbb{C}} \Rightarrow S_{\mathbb{C}}$ *with the predicate* $f \mapsto \Pi_{r:\widehat{R}} S_{\mathbb{P}}(f[r_{\mathbb{C}}])$, *where the* Π*-type is once again seen as a type in* $\mathcal{U}$.

We have the obvious strictly cartesian carrier functor $\mathbb{C}(\mathbf{A},\mathcal{U}) : \mathbb{P}(\mathbf{A},\mathcal{U}) \to \mathbf{A}$ *into the original opfibration.*

We are now ready for our main construction. Since the definition of a predicate opfibration is with respect to a given subuniverse, we parametrize our definition over a given *family* of subuniverses, one for each $l \leq p$, telling us what kind of predicates to use at level l. We then instantiate our definition by putting $\mathcal{U}_0 \coloneqq \mathcal{U}_{\mathsf{all}}$ when $p = 0$ and, when $p > 0$, putting

$$\mathcal{U}_l = \begin{cases} \mathcal{U}_{\mathsf{all}} & \text{if } l < p \\ \mathcal{U}_{\mathsf{prop}} & \text{if } l = p \end{cases}$$

We recall that the carrier of a 2-relation is a quadruple of 1-relations forming a square. Generalizing this idea, we want the carrier of a 3-relation to be a sextuple of 2-relations forming a cube, *etc.* To express this *boundary condition* of "forming a (higher) square", we already need to have face maps defined at the previous level. Thus, we start by constructing a hierarchy of the following form in eCIC:



Here, for each l, $\mathbf{A}_l$ and $\mathbf{B}_l$ are cartesian closed opfibrations and $\mathbf{C}_l$ is a strictly cartesian functor. Moreover, F, f are eCIC families of strictly cartesian functors, one for each $l, k : \mathbb{N}$ such that $k \leq l < p$, and $\star : 2$, with $\mathsf{F}_l(k,\star) : \mathsf{A} + l + 1 \to \mathsf{B}_l$ and $\mathsf{f}_l(k,\star) : \mathsf{B}_{l+1} \to \mathsf{B}_l$, subject to the commutativity condition $\mathsf{f}_l(k,\star) = \mathsf{F}_l(k,\star) \circ \mathsf{C}_{l+1}$.

The opfibrations $\mathbf{A}_l$ are the carriers of relations at level l, with $\mathsf{Rel}(l) \coloneqq \mathbf{B}_l$ constructed as a predicate opfibration from the corresponding $\mathbf{A}_l$ and $\mathbf{C}_l$ the associated carrier functor. The face maps $\mathsf{f}_l(k,\star)$ extract the specified face from a relation at level $l+1$, with $\mathsf{F}_l(k,\star)$ acting only on carriers.

We proceed by induction on $l \leq p$. For the base case $l = 0$, we define $\mathbf{A}_0 \coloneqq \mathbf{1}$ and $\mathbf{B}_0 \coloneqq \mathbb{P}(\mathbf{A}_0, \mathcal{U}_0)$. A relation at level 0 thus pairs the unique object in $\mathbf{1}$ – *i.e.*, the canonical term of 1 – with a proof-relevant predicate A on its elements. Since the type of elements of the unique object in $\mathbf{1}$ is again 1, we have $A : 1 \to \mathcal{U}_{\mathsf{all}}$; in particular relations at level 0 are just sets $A : \mathbb{U}$. An element of A pairs the canonical term of 1 with a witness $a : A$ for the associated predicate, so elements of the 0-relation A are just terms of A. A morphism from A to B pairs the canonical element of 1 with a map $f : 1 \times A \to 1 \times B$, so morphisms of 0-relations are just functions $f : A \to B$ between sets and the mapping $\widehat{f}$ is just f.

For the inductive case, assume all levels up to and including the level l of the hierarchy have been constructed. Let Ob and Mor be the objects and morphisms of $\mathbf{B}_l$, respectively. We consider the zero case $l = 0$ and the successor case $l > 0$ separately. For the former, we define the carrier opfibration $\mathbf{A}_1$ as follows:

- objects are maps $X : 2 \to \mathsf{Ob}$
- morphisms from X to Y are maps $M : 2 \to \mathsf{Mor}$ with $M(\star)$ a morphism from $X(\star)$ to $Y(\star)$
- identity and composition of morphisms are defined pointwise
- for an object X, $\widehat{X} \coloneqq \Pi_{\star:2}\,\widehat{X(\star)}$
- for a morphism M from X to Y, $(\widehat{M}\,x)\,\star \coloneqq \widehat{M(\star)}\,x(\star)$
- terminal objects, products, and exponentials are defined pointwise

Put $\mathbf{B}_1 \coloneqq \mathbb{P}(\mathbf{A}_1, \mathcal{U}_1)$. Define $\mathsf{F}_0(0,\star) : \mathbf{A}_1 \to \mathbf{B}_0$ for $\star : 2$ to be the following functor:

- on objects, we have $\mathsf{F}_0(0,\star)\,X \coloneqq X(\star)$
- on morphisms, we have $\mathsf{F}_0(0,\star)\,M \coloneqq M(\star)$
- on elements, we have $\mathsf{F}_0(0,\star)_X\,x \coloneqq x(\star)$

Finally, we put $\mathsf{f}_0(0,\star) \coloneqq \mathsf{F}_0(0,\star) \circ \mathbb{C}(\mathsf{A}_1, \mathcal{U}_1)$.

So the carriers of relations at level 1 are pairs of sets $(R_\top, R_\bot)$, and elements of $(R_\top, R_\bot)$ are pairs of terms $(r_\top, r_\bot) : R_\top \times R_\bot$. Morphisms from $(R_\top, R_\bot)$ to $(S_\top, S_\bot)$ are pairs $(f_\top, f_\bot) : (R_\top \to S_\top) \times (R_\bot \to S_\bot)$ of functions, with $\widehat{(f_\top, f_\bot)}$ carrying $(r_\top, r_\bot)$ to $(f_\top(r_\top), f_\bot(r_\bot))$.

Relations at level 1 are pairs of sets $(R_\top, R_\bot)$ together with a predicate R on pairs $(r_\top, r_\bot) : R_\top \times R_\bot$. An element of $((R_\top, R_\bot), R)$ is a pair $(r_\top, r_\bot) : R_\top \times R_\bot$ together with a witness $p : R(r_\top, r_\bot)$. A morphism from $((R_\top, R_\bot), R)$ to $((S_\top, S_\bot), S)$ is a pair of functions $(f_\top, f_\bot) : (R_\top \to S_\top) \times (R_\bot \to S_\bot)$ plus a map g that takes an element $((r_\top, r_\bot), p)$ to a witness $g((r_\top, r_\bot), p) : S(f_\top(r_\top), f_\bot(r_\bot))$. Finally, the mapping $\widehat{((f_\top, f_\bot), g)}$ carries an element $((r_\top, r_\bot), p)$ to the element $\big((f_\top(r_\top), f_\bot(r_\bot)), g((r_\top, r_\bot), p)\big)$.

The face maps $\mathsf{f}_0(0,\top)$ and $\mathsf{f}_0(0,\bot)$, which in our running examples we called $\mathsf{f}_\top$ and $\mathsf{f}_\bot$, select the specified face from the carrier of any object, morphism, or element.

For the successor case, we define the carrier opfibration $\mathbf{A}_{l+2}$ for $l : \mathbb{N}$ as follows:

- Objects are maps $X : \Pi_{k:\mathbb{N},\, k\leq l+1}\, 2 \to \mathsf{Ob}$ subject to a *boundary condition*, which says that for any $i, j : \mathbb{N}$ with $i \leq j \leq l$ and any $\star_1, \star_2 : 2$ we have

$$\mathsf{f}_l(i,\star_1)\, X(j+1,\star_2) = \mathsf{f}_l(j,\star_2)\, X(i,\star_1)$$

  This condition corresponds to the cubical equation

$$\mathsf{f}_l(i,\star_1) \circ \mathsf{f}_{l+1}(j+1,\star_2) = \mathsf{f}_l(j,\star_2) \circ \mathsf{f}_{l+1}(i,\star_1)$$

  if $i \leq j$ and expresses formally the earlier requirement of "forming a (higher) square".

- Morphisms from X to Y are maps $M : \Pi_{k:\mathbb{N},\, k\leq l+1}\, 2 \to \mathsf{Mor}$ with $M(k,\star)$ a morphism from $X(k,\star)$ to $Y(k,\star)$, subject to the same boundary condition

$$\mathsf{f}_l(i,\star_1)\, M(j+1,\star_2) = \mathsf{f}_l(j,\star_2)\, M(i,\star_1)$$

  for any $i, j : \mathbb{N}$ with $i \leq j \leq l$ and $\star_1, \star_2 : 2$.

- Identity and composition of morphisms are defined pointwise.

- Elements of X are functions $x : \Pi_{k:\mathbb{N},\, k\leq l+1} \Pi_{\star:2} \widehat{X(k,\star)}$, once again subject to a similar boundary condition

$$\mathsf{f}_l(i,\star_1)\, x(j+1,\star_2) = \mathsf{f}_l(j,\star_2)\, x(i,\star_1)$$

  for any $i, j : \mathbb{N}$ with $i \leq j \leq l$ and any $\star_1, \star_2 : 2$. The above equality type-checks due to the fact that X itself satisfies the corresponding boundary condition.

- For a morphism M from X to Y, we put

$$(\widehat{M}\, x)\,(k,\star) \coloneqq \widehat{M(k,\star)}\, x(k,\star)$$

- Terminal objects, products, and exponentials are defined pointwise.

Put $\mathbf{B}_{l+2} \coloneqq \mathbb{P}\big(\mathbf{A}_{l+2}, \mathcal{U}_{l+2}\big)$ and define $\mathsf{F}_{l+1}(k,\star) : \mathbf{A}_{l+2} \to \mathbf{B}_{l+1}$ for any $k : \mathbb{N}$ with $k \leq l+1$ and $\star : 2$ as follows:

- on objects, we have $\mathsf{F}_{l+1}(k,\star)\, X \coloneqq X(k,\star)$
- on morphisms, we have $\mathsf{F}_{l+1}(k,\star)\, M \coloneqq M(k,\star)$
- on elements, we have $\mathsf{F}_{l+1}(k,\star)_X\, x \coloneqq x(k,\star)$

We put $\mathsf{f}_{l+1}(k,\star) \coloneqq \mathsf{F}_{l+1}(k,\star) \circ \mathbb{C}\big(\mathsf{A}_{l+2}, \mathcal{U}_{l+2}\big)$.

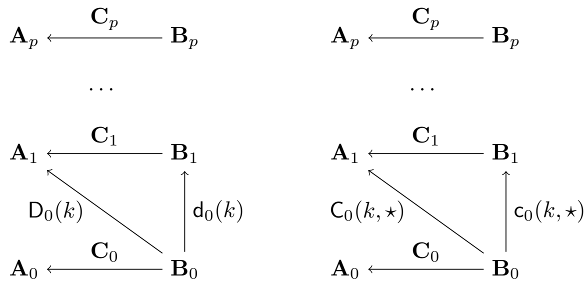
So at levels $l \geq 2$, the carriers are again tuples of objects from the previous level, one for each face, *e.g.*, at level 2 the carriers are quadruples and at level 3 they are sextuples. Unlike at level 1, however, we need to impose the boundary condition to ensure that the endpoints properly align. For instance, when $l = 2$, this condition can be reformulated as follows:

$$\Pi_{\star_1:2}\Pi_{\star_2:2}\, \mathsf{f}_0(0,\star_1)\, X(1,\star_2) = \mathsf{f}_0(0,\star_2)\, X(0,\star_1)$$

Recalling that $\mathsf{f}_0(0,\top)$ and $\mathsf{f}_0(0,\bot)$ stand for the domain and codomain projections, respectively, and letting $Q_{k\star}$ denote $X(k,\star)$, the boundary condition then says precisely that the domains of $Q_{1\top}$ and $Q_{0\top}$ agree, the domain of $Q_{1\bot}$ agrees with the codomain of $Q_{0\top}$, the codomain of $Q_{1\top}$ agrees with the domain of $Q_{0\bot}$, and the codomains of $Q_{1\bot}$ and $Q_{0\bot}$ agree, so the quadruple indeed forms a square. We get the four face maps $\mathsf{f}_1(0,\top)$, $\mathsf{f}_1(0,\bot)$, $\mathsf{f}_1(1,\top)$, $\mathsf{f}_1(1,\bot)$, which we previously called $\mathsf{f}_{0\top}$, $\mathsf{f}_{0\bot}$, $\mathsf{f}_{1\top}$, $\mathsf{f}_{1\bot}$, each selecting the given face from the carrier of an object, morphism, or element.

Note carefully that since the boundary condition needs to be part of the syntactic type of carriers at level $l+2$, it is crucial that f_l be an eCIC term itself.

To define degeneracies, we extend our previous hierarchy by two eCIC families D and d of cartesian functors, one for each $l, k : \mathbb{N}$ such that $k \leq l < p$, with $\mathsf{D}_l(k) : \mathsf{B}(l) \to \mathsf{A}(l+1)$ and $\mathsf{d}_l(k) : \mathsf{B}(l) \to \mathsf{A}(l+1)$, subject to the commutativity condition $\mathsf{d}_l(k) = \mathsf{D}_l(k) \circ \mathsf{C}(l+1)$ and the requirement that D and F (and hence d and f) satisfy the cubical equalities on the exchange of face maps and degeneracies. To define connections, we further extend this hierarchy by two eCIC families C and c of cartesian functors, one for each $l, k : \mathbb{N}$ such that $k < l < p$ and $\star : 2$, with $\mathsf{C}_l(k,\star) : \mathsf{B}(l) \to \mathsf{A}(l+1)$ and $\mathsf{c}_l(k,\star) : \mathsf{B}(l) \to \mathsf{A}(l+1)$, satisfying the commutativity condition $\mathsf{c}_l(k,\star) = \mathsf{D}_l(k,\star) \circ \mathsf{C}(l+1)$ and the requirement that C and F (and hence c and f) satisfy the cubical equalities on the exchange of face maps and connections.



Furthermore, we will maintain the following invariants:

- $\mathbf{C}_l\,(\mathsf{F}_l(k,\top)\,x) = \mathbf{C}_l\,(\mathsf{F}_l(k,\bot)\,x)$ if $x : \widehat{\mathsf{D}_l(k)\,A}$
- $\mathbf{C}_l\,(\mathsf{F}_l(k,\star)\,x) = \mathbf{C}_l\,(\mathsf{F}_l(k+1,\star)\,x)$ if $x : \widehat{\mathsf{C}_l(k,\star)\,A}$

In other words, the faces $\mathsf{F}_l(k,\top)\,x$ and $\mathsf{F}_l(k,\bot)\,x$ of an element $x : \widehat{\mathsf{D}_l(k)\,A}$ have the same carriers, even though their predicate parts might differ. Similarly for the faces $\mathsf{F}_l(k,\star)\,x$ and $\mathsf{F}_l(k+1,\star)\,x$ of an element $x : \widehat{\mathsf{C}_l(k,\star)\,A}$. The above invariants are established in tandem with the following:

- $\mathsf{f}_l(k,\top)\,x = \mathsf{f}_l(k,\bot)\,x$ if $x : \widehat{\mathsf{d}_l(k)\,A}$
- $\mathsf{f}_l(k,\star)\,x = \mathsf{f}_l(k+1,\star)\,x$ if $x : \widehat{\mathsf{c}_l(k,\star)\,A}$

We construct $\mathsf{D}, \mathsf{d}, \mathsf{C}, \mathsf{c}$ by induction on $l \le p$. For $l = 0$, we define $\mathsf{D}_0(0) : \mathbf{B}(0) \to \mathbf{A}(1)$ as follows:

- on objects, $\mathsf{D}_0(0)\,A$ is the constant map $\star \mapsto A$
- on morphisms, $\mathsf{D}_0(0)\,M$ is the constant map $\star \mapsto M$
- on elements, $\mathsf{D}_0(0)_A\,a$ is the constant map $\star \mapsto a$

We define the functor $\mathsf{d}_0(0) : \mathbf{B}(0) \to \mathbf{B}(1)$ as follows:

- On objects, $\mathsf{d}_0(0)\,A$ pairs up the carrier $\mathsf{D}_0(0)\,A$ with the predicate $x \mapsto x(\top)_{\mathbb{P}} = x(\bot)_{\mathbb{P}}$, where the identity type is seen as a term of $\mathcal{U}_1$ and is well-formed by one of the aforementioned invariants.
- On elements $a \coloneqq (x,p) : \widehat{A}$, $\mathsf{d}_0(0)\,a$ pairs up the carrier $\mathsf{D}_0(0)\,a$ with the reflexivity proof on p.
- On morphisms $M : A \to B$, $\mathsf{d}_0(0)\,M$ has the morphism $\mathsf{D}_0(0)\,M$ as the carrier. We must endow it with a function $g : \widehat{\mathsf{d}_0(0)\,A} \to \widehat{\mathsf{d}_0(0)\,B}$ that is compatible with the carrier. If it exists, such a map is necessarily unique. Let $(x,p) : \widehat{\mathsf{d}_0(0)\,A}$. Then $x(\top)_{\mathbb{P}} = x(\bot)_{\mathbb{P}}$ as witnessed by p, which implies $x(\top) = x(\bot)$. Since g is compatible with $\mathsf{D}_0(0)\,M$, the carrier of $g(x,p)$ must be $\widehat{\mathsf{D}_0(0)\,M}\,x$. So it suffices to show that

  $$(\widehat{\mathsf{D}_0(0)\,M}\,x)(\top) = (\widehat{\mathsf{D}_0(0)\,M}\,x)(\bot)$$

  Since $(\widehat{\mathsf{D}_0(0)\,M}\,x)(\star) = \widehat{M}\,x(\star)$, we just need to show

  $$\widehat{M}\,x(\top) = \widehat{M}\,x(\bot)$$

  But this is obvious since $x(\top) = x(\bot)$.

Since in the base case the type $k < l$ is empty, there are no connections to construct and the families C, c are vacuous.

So the degeneracy $\mathsf{d}_0(0)$, which we earlier called d, duplicates a set A as both the top and the bottom face, and requires that the two faces of any element $x : \widehat{\mathsf{D}_0(0,\star)\,A}$ coincide.

For the inductive case, fix $l \in \mathbb{N}$. Let $\mathsf{D}_{l+1}(k) : \mathsf{Rel}(l+1) \to \mathbf{A}(l+2)$ for $k : \mathbb{N}$ with $k \le l+1$ be the following functor:

- On objects, $\mathsf{D}_{l+1}(k)\,A$ is the map

$$(j,\star) \mapsto \begin{cases} A & \text{if } j = k \\ \mathsf{d}_l(k')\ \mathsf{f}_l(j,\star)\ A & \text{if } j < k \coloneqq k'+1 \\ \mathsf{d}_l(k)\ \mathsf{f}_l(j',\star)\ A & \text{if } k < j \coloneqq j'+1 \end{cases}$$

  motivated by the three cubical equations on the exchange of face maps and degeneracies.

- On morphisms, $\mathsf{D}_{l+1}(k)\,M$ is the map

$$(j,\star) \mapsto \begin{cases} M & \text{if } j = k \\ \mathsf{d}_l(k')\ \mathsf{f}_l(j,\star)\ M & \text{if } j < k \coloneqq k'+1 \\ \mathsf{d}_l(k)\ \mathsf{f}_l(j',\star)\ M & \text{if } k < j \coloneqq j'+1 \end{cases}$$

- On elements, $\mathsf{D}_{l+1}(k)_A\,a$ is the map

$$(j,\star) \mapsto \begin{cases} a & \text{if } j = k \\ \mathsf{d}_l(k')\ \mathsf{f}_l(j,\star)\ a & \text{if } j < k \coloneqq k'+1 \\ \mathsf{d}_l(k)\ \mathsf{f}_l(j',\star)\ a & \text{if } k < j \coloneqq j'+1 \end{cases}$$

We define the functor $\mathsf{d}_{l+1}(k) : \mathbf{B}_{l+1} \to \mathbf{B}_{l+2}$ for $k : \mathbb{N}$ with $k \leq l+1$ as follows:

- On objects, $\mathsf{d}_{l+1}(k)\,A$ pairs up the carrier $\mathsf{D}_{l+1}(k)\,A$ with the map $x \mapsto x(k,\top)_{\mathbb{P}} = x(k,\bot)_{\mathbb{P}}$, where the identity is seen as a type in $\mathcal{U}_{l+2}$.
- On elements $a \coloneqq (x,p) : \widehat{A}$, $\mathsf{d}_{l+1}(k)\,a$ endows the carrier $\mathsf{D}_{l+1}(k)\,a$ with the reflexivity proof on p.
- On morphisms $M : A \to B$, the morphism $\mathsf{d}_{l+1}(k)\,M$ endows the carrier $\mathsf{D}_{l+1}(k)\,M$ with the unique function $g : \widehat{\mathsf{d}_{l+1}(k)\,A} \to \widehat{\mathsf{d}_{l+1}(k)\,B}$ compatible with the carrier, constructed analogously to the base case.

We define the functor $\mathsf{C}_{l+1}(k,\star) : \mathbf{B}_{l+1} \to \mathbf{A}_{l+2}$ for $k : \mathbb{N}$ with $k < l+1$ and $\star : 2$ as follows:

- on objects, $\mathsf{C}_{l+1}(k,\star)\,A$ is the map

$$(j,\star') \mapsto \begin{cases} A & \text{if } j = k \text{ and } \star' = \star \\ A & \text{if } j = k+1 \text{ and } \star' = \star \\ \mathsf{d}_l(k)\,\mathsf{f}_l(k,\overline{\star})\,A & \text{if } j = k \text{ and } \star' = \overline{\star} \\ \mathsf{d}_l(k)\,\mathsf{f}_l(k,\overline{\star})\,A & \text{if } j = k+1 \text{ and } \star' = \overline{\star} \\ \mathsf{c}_l(k',\star)\,\mathsf{f}_l(j,\star')\,A & \text{if } j < k \coloneqq k'+1 \\ \mathsf{c}_l(k,\star)\,\mathsf{f}_l(j',\star')\,A & \text{if } k+1 < j \coloneqq j'+1 \end{cases}$$

  motivated by the six cubical equations on the exchange of face maps and connections.

- On morphisms, $\mathsf{C}_{l+1}(k,\star)\,M$ is the map

$$(j,\star') \mapsto \begin{cases} M & \text{if } j = k \text{ and } \star' = \star \\ M & \text{if } j = k+1 \text{ and } \star' = \star \\ \mathsf{d}_l(k)\,\mathsf{f}_l(k,\overline{\star})\,M & \text{if } j = k \text{ and } \star' = \overline{\star} \\ \mathsf{d}_l(k)\,\mathsf{f}_l(k,\overline{\star})\,M & \text{if } j = k+1 \text{ and } \star' = \overline{\star} \\ \mathsf{c}_l(k',\star)\,\mathsf{f}_l(j,\star')\,M & \text{if } j < k \coloneqq k'+1 \\ \mathsf{c}_l(k,\star)\,\mathsf{f}_l(j',\star')\,M & \text{if } k+1 < j \coloneqq j'+1 \end{cases}$$

- On elements, $\mathsf{C}_{l+1}(k,\star)_A\,a$ is the map

$$(j,\star') \mapsto \begin{cases} a & \text{if } j = k \text{ and } \star' = \star \\ a & \text{if } j = k+1 \text{ and } \star' = \star \\ \mathsf{d}_l(k)\,\mathsf{f}_l(k,\overline{\star})\,a & \text{if } j = k \text{ and } \star' = \overline{\star} \\ \mathsf{d}_l(k)\,\mathsf{f}_l(k,\overline{\star})\,a & \text{if } j = k+1 \text{ and } \star' = \overline{\star} \\ \mathsf{c}_l(k',\star)\,\mathsf{f}_l(j,\star')\,a & \text{if } j < k \coloneqq k'+1 \\ \mathsf{c}_l(k,\star)\,\mathsf{f}_l(j',\star')\,a & \text{if } k+1 < j \coloneqq j'+1 \end{cases}$$

We define the functor $\mathsf{c}_{l+1}(k,\star) : \mathbf{B}_{l+1} \to \mathbf{B}_{l+2}$ for $k : \mathbb{N}$ with $k \leq l+1$ and $\star : 2$ as follows:

- On objects, $\mathsf{c}_{l+1}(k,\star)\,A$ pairs up the carrier $\mathsf{C}_{l+1}(k,\star)\,A$ with the map $x \mapsto x(k,\star)_{\mathbb{P}} = x(k+1,\star)_{\mathbb{P}}$, where the identity is seen as a type in $\mathcal{U}_{l+2}$.
- On elements $a \coloneqq (x,p) : \widehat{A}$, $\mathsf{c}_{l+1}(k,\star)\,a$ pairs the carrier $\mathsf{C}_{l+1}(k,\star)\,a$ with the reflexivity proof on p.
- On morphisms $M : A \to B$, the morphism $\mathsf{c}_{l+1}(k,\star)\,M$ endows the carrier $\mathsf{C}_{l+1}(k,\star)\,M$ with the unique map $g : \widehat{\mathsf{c}_{l+1}(k,\star)\,A} \to \widehat{\mathsf{c}_{l+1}(k,\star)\,B}$ compatible with the carrier, constructed analogously to the base case.

This finishes the construction of degeneracies and connections. The degeneracies $\mathsf{d}_1(0)$ and $\mathsf{d}_1(1)$, previously called $\mathsf{d}_=$ and $\mathsf{d}_\parallel$, and connections $\mathsf{c}_1(0,\top)$ and $\mathsf{c}_1(0,\bot)$, previously called $\mathsf{c}_\top$ and $\mathsf{c}_\bot$, all duplicate a relation R at the appropriate opposing (resp., adjacent) faces, and require that the corresponding faces of any element x of the carrier coincide.

Showing that the remaining cubical equalities hold is now simple. By construction,

$$\mathsf{f}_l(i,\star_1) \circ \mathsf{F}_{l+1}(j+1,\star_2) = \mathsf{f}_l(j,\star_2) \circ \mathsf{F}_{l+1}(i,\star_1)$$

if $i \leq j$, which implies that f satisfies the cubical equality for the exchange of face maps. The cubical equality

$$\mathsf{d}_{l+1}(i+1) \circ \mathsf{d}_l(j) = \mathsf{d}_{l+1}(j) \circ \mathsf{d}_l(i) \quad \text{if } j \leq i$$

is established by induction on $l \leq p$, simultaneously with

$$\mathsf{D}_{l+1}(i+1) \circ \mathsf{d}_l(j) = \mathsf{D}_{l+1}(j) \circ \mathsf{d}_l(i) \quad \text{if } j \leq i$$

The latter follows right away from the inductive hypothesis since objects in the carrier obfibrations are completely determined by their faces. To establish the former, we recall that the predicate part of a degeneracy is a family of identity types, which are proof-irrelevant. Hence the only real work is showing that the two compositions agree on objects. Fix an object A in $\mathbf{B}(l)$. The latter equality shows that the carriers of $\mathsf{d}_{l+1}(i{+}1)\,(\mathsf{d}_l(j)\,A)$ and $\mathsf{d}_{l+1}(j)\,(\mathsf{d}_l(i)\,A)$ coincide, so it only remains to show that their predicate parts coincide, *i.e.*, that they assign the same proposition to any element x of the carrier. On the left-hand side we have the proposition $x(i+1,\top)_{\mathbb{P}} = x(i+1,\bot)_{\mathbb{P}}$ whereas on the right hand side we have $x(j,\top)_{\mathbb{P}} = x(j,\bot)_{\mathbb{P}}$. The identity type on the left-hand side is indexed by the type $\mathsf{f}_l(j,\top)\,x(i+1,\top) = \mathsf{f}_l(j,\bot)\,x(i+1,\bot)$ and on the right-hand side by $\mathsf{f}_l(i,\top)\,x(j,\top) = \mathsf{f}_l(i,\bot)\,x(j,\bot)$. But by the boundary condition on x, these are one and the same, which in particular forces the left- and right-hand side propositions to coincide too as the identity type is proof-irrelevant. The other cubical equalities follow analogously.

We have thus completed our construction of a p-dimensional cartesian closed cubical category for every $p \in \mathbb{N}$. Choosing all isomorphisms to be relevant, just like we did in Example 13 thus yields a p-dimensional cartesian closed cubical category with isomorphisms $\mathsf{Iso} \hookrightarrow \mathsf{Rel}$ for every $p \in \mathbb{N}$.

7 Examples: p-Parametric Models for $p \leq 3$

We can now describe the adjunctions for the canonical cartesian closed fibrations with isomorphisms induced by $\mathsf{Iso} \hookrightarrow \mathsf{Rel}$ at dimensions $p = 1,2,3$. At each level, we only give the predicate parts of $(\forall_n \mathcal{F})(l)\,\overline{R}$ since the carriers are fully determined by the preservation of face maps.

We start with Reynolds' model. On level $l = 0$, the predicate part of $\forall_n\,\mathcal{F}(0)\,\overline{R}$ maps an element $\star : 1$ to the type below:

$$\begin{aligned}
\big\{&\phi^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(0)_0} \widehat{\mathcal{F}(0)(\overline{R},R)}\ \& \\
&\phi^{\mathsf{id}}_{0} : \Pi_{R:\mathsf{Rel}(1)_0} \widehat{\mathcal{F}(1)\big(\mathsf{d}_0(0)^n\,\overline{R},R\big)}\ \& \\
&\Pi_{i:\mathsf{Iso}(0)_1} \widehat{\mathcal{F}(0)\,(\mathsf{id},i)}\,\phi^{\mathsf{id}}_{\mathsf{id}}(\mathsf{s}(i)) = \phi^{\mathsf{id}}_{\mathsf{id}}(\mathsf{t}(i))\ \& \\
&\Pi_{i:\mathsf{Iso}(1)_1} \widehat{\mathcal{F}(1)\,(\mathsf{id},i)}\,\phi^{\mathsf{id}}_{0}(\mathsf{s}(i)) = \phi^{\mathsf{id}}_{0}(\mathsf{t}(i))\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\top)\,\phi^{\mathsf{id}}_{0}(R) = \phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\top)\,R\big)\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\bot)\,\phi^{\mathsf{id}}_{0}(R) = \phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\bot)\,R\big)\big\}
\end{aligned}$$

Here we depart significantly from the usual presentation to make the generalization to dimensions 2 and 3 easier. The content of the above type can be divided into three parts.

• To each level $l \leq k \leq p$ and each *derived* functor $M : \mathsf{Rel}(l) \to \mathsf{Rel}(k)$ we associate $\phi : \Pi_{R:\mathsf{Rel}(k)_0} \widehat{\mathcal{F}(k)\,(M^n\,\overline{R}, R)}$. A derived functor is one arising as a composition of degeneracies and/or connections. Hence, we account for all possible ways to turn the relations $\overline{R}$ in $\mathsf{Rel}(l)$ into relations in $\mathsf{Rel}(k)$. In our case, $p = 1$ and $l = 0$ so we only have to account for the identity $\mathsf{id} : \mathsf{Rel}(0) \to \mathsf{Rel}(0)$ and the single degeneracy $\mathsf{d}_0(0) : \mathsf{Rel}(0) \to \mathsf{Rel}(1)$. The former corresponds to the map $\phi^{\mathsf{id}}_{\mathsf{id}}$ and the latter to ϕ^{id}_0, where the subscript 0 indicates we take the degeneracy in dimension 0 and the superscript is reserved for connections, which only arise at higher levels.
• The second part establishes that the maps $\phi^{\mathsf{id}}_{\mathsf{id}}$, ϕ^{id}_0 are compatible with the functorial action of $\mathcal{F}$ in the following sense: given an isomorphism $i : \mathsf{Iso}(0)_1$ from A to B, $\phi^{\mathsf{id}}_{\mathsf{id}}(A)$ is an element of the relation $\mathcal{F}(0)(\overline{R}, A)$ and $\phi^{\mathsf{id}}_{\mathsf{id}}(B)$ of the relation $\mathcal{F}(0)(\overline{R}, B)$. Since $\mathcal{F}(0)$ is a functor, we have a canonical isomorphism between these two relations, and we request that it takes $\phi^{\mathsf{id}}_{\mathsf{id}}(A)$ precisely to $\phi^{\mathsf{id}}_{\mathsf{id}}(B)$. Similarly for $\phi^{\mathsf{id}}_{\mathsf{id}}$, although in this case the condition follows automatically since the predicates at the highest level are proof-irrelevant, but we include it for uniformity.
• The third part ensures that $\phi^{\mathsf{id}}_{\mathsf{id}}, \phi^{\mathsf{id}}_0$ return elements with the correct endpoints. Since there are no face maps out of $\mathsf{Rel}(0)$, there is nothing to do for $\phi^{\mathsf{id}}_{\mathsf{id}}$. In case of ϕ^{id}_0, there are two endpoints to characterize, one for each face map $\mathsf{f}_0(0,\star)$ out of $\mathsf{Rel}(1)$. We have $\mathsf{f}_0(0,\star) \circ \mathsf{d}_0(0) = \mathsf{id}$, which means that the endpoint of $\phi^{\mathsf{id}}_0(R)$ selected by $\mathsf{f}_0(0,\star)$ is an element of $\mathcal{F}(0)\,\big(\overline{R}, \mathsf{f}_0(0,\star)\,R\big)$. We have another way of producing an element of this relation, namely $\phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\star)\,R\big)$, and hence we require that these coincide.

We use the same idea at level $l = 1$. The predicate part of $\forall_n\,\mathcal{F}(1)\,\overline{R}$ maps an element X of the carrier to the type

$$
\begin{aligned}
&\big\{\theta^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(1)_0} \widehat{\mathcal{F}(1)(\overline{R}, R)}\ \&\\
&\ \Pi_{i:\mathsf{Iso}(1)_1} \widehat{\mathcal{F}(1)\,(\mathsf{id}, i)}\,\theta^{\mathsf{id}}_{\mathsf{id}}(\mathsf{s}(i)) = \theta^{\mathsf{id}}_{\mathsf{id}}(\mathsf{t}(i))\ \&\\
&\ \Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\top)\,\theta^{\mathsf{id}}_{\mathsf{id}}(R) = \big(\mathsf{F}_0(0,\top)\,X\big)^{\mathsf{id}}_{\mathsf{id}}\,\big(\mathsf{f}_0(0,\top)\,R\big)\ \&\\
&\ \Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\bot)\,\theta^{\mathsf{id}}_{\mathsf{id}}(R) = \big(\mathsf{F}_0(0,\bot)\,X\big)^{\mathsf{id}}_{\mathsf{id}}\,\big(\mathsf{f}_0(0,\bot)\,R\big)\big\}
\end{aligned}
$$

We only have to account for the identity $\mathsf{id} : \mathsf{Rel}(1) \to \mathsf{Rel}(1)$ so we have a single map $\theta^{\mathsf{id}}_{\mathsf{id}}$. As before, there is a condition ensuring compatibility with the functorial action of $\mathcal{F}$. The endpoint conditions are interesting now: we have $\mathsf{f}_0(0,\star) \circ \mathsf{id} = \mathsf{id} \circ \mathsf{f}_0(0,\star)$ so, unlike in the previous case, the face map is not absorbed. This means that the endpoint of $\theta^{\mathsf{id}}_{\mathsf{id}}(R)$ selected by $\mathsf{f}_0(0,\star)$ is an element of $\mathcal{F}(0)\,\big(\mathsf{f}_0(0,\star)^n\,\overline{R}, \mathsf{f}_0(0,\star)\,R\big)$. An alternative way to produce an element of this relation is to take the $\phi^{\mathsf{id}}_{\mathsf{id}}$ component of the endpoint $(\mathsf{F}_0(0,\star)\,X)^{\mathsf{id}}_{\mathsf{id}}$ of X selected by the face map $\mathsf{F}_0(0,\star)$ and apply it to $\mathsf{f}_0(0,\star)\,R$.

We proceed analogously when generating the adjoints for $p = 2, 3$. For $p = 2$, the predicate part of $\forall_n\,\mathcal{F}(0)\,\overline{R}$ maps an element $\star : \mathbf{1}$ to the type below:

$$
\begin{aligned}
&\big\{\phi^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(0)_0} \widehat{\mathcal{F}(0)(\overline{R}, R)}\ \&\\
&\ \phi^{\mathsf{id}}_0 : \Pi_{R:\mathsf{Rel}(1)_0} \widehat{\mathcal{F}(1)\big(\mathsf{d}_0(0)^n\,\overline{R}, R\big)}\ \&\\
&\ \phi^{\mathsf{id}}_{0,0} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_1(0)^n\,\mathsf{d}_0(0)^n\,\overline{R}, R\big)}\ \&\\
&\ \Pi_{i:\mathsf{Iso}(0)_1} \widehat{\mathcal{F}(0)\,(\mathsf{id}, i)}\,\phi^{\mathsf{id}}_{\mathsf{id}}(\mathsf{s}(i)) = \phi^{\mathsf{id}}_{\mathsf{id}}(\mathsf{t}(i))\ \&\\
&\ \Pi_{i:\mathsf{Iso}(1)_1} \widehat{\mathcal{F}(1)\,(\mathsf{id}, i)}\,\phi^{\mathsf{id}}_0(\mathsf{s}(i)) = \phi^{\mathsf{id}}_0(\mathsf{t}(i))\ \&\\
&\ \Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\top)\,\phi^{\mathsf{id}}_0(R) = \phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\top)\,R\big)\ \&\\
&\ \Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\bot)\,\phi^{\mathsf{id}}_0(R) = \phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\bot)\,R\big)\ \&\\
&\ \ldots\big\}
\end{aligned}
$$

where the corresponding functoriality and endpoint conditions for $\phi^{\mathsf{id}}_{0,0}$ are as expected. The predicate part of

$\forall_n \mathcal{F}(1)\, \overline{R}$ maps an element X of the carrier to the type

$$\begin{aligned}
\{&\theta^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(1)_0} \widehat{\mathcal{F}(1)(\overline{R}, R)}\ \& \\
&\theta^{\mathsf{id}}_{0} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_1(0)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{\mathsf{id}}_{1} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_1(1)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\Pi_{i:\mathsf{Iso}(1)_1} \widehat{\mathcal{F}(1)\,(\mathsf{id}, i)}\, \theta^{\mathsf{id}}_{\mathsf{id}}(\mathsf{s}(i)) = \theta^{\mathsf{id}}_{\mathsf{id}}(\mathsf{t}(i))\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\top)\, \theta^{\mathsf{id}}_{\mathsf{id}}(R) = \big(\mathsf{F}_0(0,\top)\, X\big)^{\mathsf{id}}_{\mathsf{id}} \big(\mathsf{f}_0(0,\top)\, R\big)\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\bot)\, \theta^{\mathsf{id}}_{\mathsf{id}}(R) = \big(\mathsf{F}_0(0,\bot)\, X\big)^{\mathsf{id}}_{\mathsf{id}} \big(\mathsf{f}_0(0,\bot)\, R\big)\ \& \\
&\ldots\}
\end{aligned}$$

The predicate part of $\forall_n \mathcal{F}(2)\, \overline{R}$ maps an element X of the carrier to the type

$$\{\sigma^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)(\overline{R}, R)}\ \& \ldots\}$$

Finally, for $p = 3$, the predicate part of $\forall_n \mathcal{F}(0)\, \overline{R}$ maps an element $\star : \mathsf{1}$ to the type below:

$$\begin{aligned}
\{&\phi^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(0)_0} \widehat{\mathcal{F}(0)(\overline{R}, R)}\ \& \\
&\phi^{\mathsf{id}}_{0} : \Pi_{R:\mathsf{Rel}(1)_0} \widehat{\mathcal{F}(1)\big(\mathsf{d}_0(0)^n\, \overline{R}, R\big)}\ \& \\
&\phi^{\mathsf{id}}_{0,0} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_1(0)^n\, \mathsf{d}_0(0)^n\, \overline{R}, R\big)}\ \& \\
&\phi^{\mathsf{id}}_{0,0,0} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(0)^n\, \mathsf{d}_1(0)^n\, \mathsf{d}_0(0)^n\, \overline{R}, R\big)}\ \& \\
&\Pi_{i:\mathsf{Iso}(0)_1} \widehat{\mathcal{F}(0)\,(\mathsf{id}, i)}\, \phi^{\mathsf{id}}_{\mathsf{id}}(\mathsf{s}(i)) = \phi^{\mathsf{id}}_{\mathsf{id}}(\mathsf{t}(i))\ \& \\
&\Pi_{i:\mathsf{Iso}(1)_1} \widehat{\mathcal{F}(1)\,(\mathsf{id}, i)}\, \phi^{\mathsf{id}}_{0}(\mathsf{s}(i)) = \phi^{\mathsf{id}}_{0}(\mathsf{t}(i))\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\top)\, \phi^{\mathsf{id}}_{0}(R) = \phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\top)\, R\big)\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\bot)\, \phi^{\mathsf{id}}_{0}(R) = \phi^{\mathsf{id}}_{\mathsf{id}}\big(\mathsf{f}_0(0,\bot)\, R\big)\ \& \\
&\ldots\}
\end{aligned}$$

The predicate part of $\forall_n\, \mathcal{F}(1)\, \overline{R}$ maps an element X of the carrier to the type

$$
\begin{aligned}
\{&\theta^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(1)_0} \widehat{\mathcal{F}(1)(\overline{R}, R)}\ \& \\
&\theta^{\mathsf{id}}_{0} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_1(0)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{\mathsf{id}}_{1} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_1(1)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{\mathsf{id}}_{0,0} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(0)^n\, \mathsf{d}_1(0)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{\mathsf{id}}_{1,0} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(0)^n\, \mathsf{d}_1(1)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{\mathsf{id}}_{1,1} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(1)^n\, \mathsf{d}_1(1)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top}_{0} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(0)^n\, \mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top}_{1} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(1)^n\, \mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top}_{2} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(2)^n\, \mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot}_{0} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(0)^n\, \mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot}_{1} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(1)^n\, \mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot}_{2} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(2)^n\, \mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top,0\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(0,\bot)^n\, \mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top,1\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(1,\top)^n\, \mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\top,1\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(1,\bot)^n\, \mathsf{c}_1(0,\top)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot,0\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(0,\top)^n\, \mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot,1\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(1,\top)^n\, \mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\theta^{0\bot,1\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(1,\bot)^n\, \mathsf{c}_1(0,\bot)^n\, \overline{R}, R\big)}\ \& \\
&\Pi_{i:\mathsf{Iso}(1)_1} \widehat{\mathcal{F}(1)\,(\mathsf{id}, i)}\, \theta^{\mathsf{id}}_{\mathsf{id}}(\mathsf{s}(i)) = \theta^{\mathsf{id}}_{\mathsf{id}}(\mathsf{t}(i))\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\top)\, \theta^{\mathsf{id}}_{\mathsf{id}}(R) = \big(\mathsf{F}_0(0,\top)\, X\big)^{\mathsf{id}}_{\mathsf{id}} \big(\mathsf{f}_0(0,\top)\, R\big)\ \& \\
&\Pi_{R:\mathsf{Rel}(1)_0} \mathsf{f}_0(0,\bot)\, \theta^{\mathsf{id}}_{\mathsf{id}}(R) = \big(\mathsf{F}_0(0,\bot)\, X\big)^{\mathsf{id}}_{\mathsf{id}} \big(\mathsf{f}_0(0,\bot)\, R\big)\ \& \\
&\ldots\}
\end{aligned}
$$

To illustrate, we give the endpoint conditions on $\theta^{0\top}_1$ corresponding to the face maps $\mathsf{f}_2(0,\star)$. We have

$$\mathsf{f}_2(0,\top) \circ \mathsf{d}_2(1) \circ \mathsf{c}_1(0,\top) = \mathsf{d}_1(0)$$

so the corresponding endpoint condition is

$$\Pi_{R:\mathsf{Rel}(3)_0} \mathsf{f}_2(0,\top)\, \theta^{0\top}_1(R) = \theta^{\mathsf{id}}_0(\mathsf{f}_2(0,\top)\, R)$$

On the other hand,

$$\mathsf{f}_2(0,\bot) \circ \mathsf{d}_2(1) \circ \mathsf{c}_1(0,\top) = \mathsf{d}_1(0) \circ \mathsf{d}_0(0) \circ \mathsf{f}_0(0,\bot)$$

so the corresponding endpoint condition is

$$\Pi_{R:\mathsf{Rel}(3)_0} \mathsf{f}_2(0,\bot)\, \theta^{0\top}_1(R) = (\mathsf{F}_0(0,\bot)\, X)_{0,0}\, (\mathsf{f}_2(0,\bot)\, R)$$

where $\big(\mathsf{F}_0(0,\bot)\, X\big)_{0,0}$ is the $\phi_{0,0}$ component of $\mathsf{F}_0(0,\bot)\, X$, accounting for the composition $\mathsf{d}_1(0) \circ \mathsf{d}_0(0)$.

The predicate part of $\forall_n\, \mathcal{F}(2)\, \overline{R}$ maps an element X of the carrier to the type

$$
\begin{aligned}
\{ & \sigma^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(2)_0} \widehat{\mathcal{F}(2)(\overline{R}, R)} \;\&\ldots\;\& \\
& \sigma^{\mathsf{id}}_{0} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(0)^n\, \overline{R}, R\big)} \;\& \\
& \sigma^{\mathsf{id}}_{1} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(1)^n\, \overline{R}, R\big)} \;\& \\
& \sigma^{\mathsf{id}}_{2} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{d}_2(2)^n\, \overline{R}, R\big)} \;\& \\
& \sigma^{0\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(0,\top)^n\, \overline{R}, R\big)} \;\& \\
& \sigma^{0\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(0,\bot)^n\, \overline{R}, R\big)} \;\& \\
& \sigma^{1\top}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(1,\top)^n\, \overline{R}, R\big)} \;\& \\
& \sigma^{1\bot}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(2)\big(\mathsf{c}_2(1,\bot)^n\, \overline{R}, R\big)} \;\& \\
& \}
\end{aligned}
$$

The predicate part of $\forall_n\, \mathcal{F}(3)\, \overline{R}$ maps an element X of the carrier to the type

$$\big\{ \tau^{\mathsf{id}}_{\mathsf{id}} : \Pi_{R:\mathsf{Rel}(3)_0} \widehat{\mathcal{F}(3)(\overline{R}, R)} \;\&\ldots \big\}$$

Corollary 31. *The models described above for $p = 1, 2, 3$ are p-parametric over* $\mathsf{Iso} \hookrightarrow \mathsf{Rel}$ *at the corresponding dimensions.*

8 Related Work

Parametricity for System F has long been of interest. Since its introduction in [25], it has been studied from a semantic perspective in, *e.g.*, [7, 10, 11, 12, 16, 20, 26, 27, 28], from a type-theoretic perspective in, *e.g.*, [3, 4, 5, 21] and from a logical perspective in, *e.g.*, [1, 23, 24, 26]. Parametricity at higher dimensions has also been studied elsewhere. In [29] we gave the proof-relevant parametric model, inspired by ideas from [11] and [22], that is the instantiation to dimension 2 of the full higher-dimensional framework we present here. In [4], cubes are used to extend pure type systems with parametricity rules so that relational interpretations of types and terms can be defined internally to the resulting extended theories. In [5], dimensions were reframed as colors. The presheaf model of [3] associated names, rather than mere indices, to colors to eliminate the explicit use of cubes, but still retains an essentially cubical structure. While the use of cubes, dimension, and (later) color in these works is similar to ours, the overall motivations and contributions are entirely different. Their goal is to develop a type theory where one can reason about parametricity internally; the aims of [21] and, more recently, [8], are similar. By contrast, we develop a framework for higher-dimensional parametricity for System F that can be instantiated both to recognize well-known lower-dimensional models as properly parametric, and, as we demonstate, to develop completely new higher-dimensional ones.

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