A modern perspective on the O’Hearn-Riecke model

Extended Abstract

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1. Introduction

Around the same time that Hyland and Ong [5, 6] and Abramsky, Jagadeesan, and Malacaria [1, 2] gave a fully abstract game semantics to PCF, O’Hearn and Riecke [14] also gave a fully abstract model combining domain-theoretic and relational techniques.

The games models give an intensional description of the semantics through a careful analysis of the interaction of a program with its environment. Once the semantics characterises the appropriate intensional interaction, one quotients the model through the extensional collapse process to get a fully abstract model. The O’Hearn-Riecke (OHR) model starts out with the usual extensional, domain-theoretic model, and then uses logical relations to cut out junk from the model. Game semantics has since been extended to deal with a wide spectrum of effects, whereas the O’Hearn-Riecke model remained relatively untouched, notably excepting Stark [16].

In the proposed talk, we will describe our ongoing work analysing the OHR model. We hope that, 25 years later, we can extend it to account for other effects. This work is at an early stage. We hope to use the workshop to stimulate informal discussion about directions for further questions, as well as learn folklore about results concerning the OHR model.

We structure our development from the modern perspective on a programming language with computational effects [10, 13]: a category for values (pre-domains), a strong monad over it, with call-by-name semantics taking place in (a suitable subcategory of) the Eilenberg-Moore category for this monad.

2. Values/pre-domains

Before presenting the value part of the OHR model, we consider a simpler construction on the category Set of sets and functions.

Example 1 (Binary endo-relations). The category ERel has as objects pairs $R = (\overline{R}, R)$ consisting of a set $\overline{R}$ and a binary endo-relation $R \subseteq \overline{R}^2$ (relation, for brevity). A morphism $f : R \to S$ is a function $f : \overline{R} \to \overline{S}$ preserving the relation:

$$(x_1, x_2) \in \overline{R} \quad \Rightarrow \quad (f(x_1), f(x_2)) \in \overline{S}$$

The category ERel is the change-of-base of the subobject fibration along the functor multiplying each set/function with itself. This category is cartesian closed, whose exponential is given as in Fig. 1.

Let $R$ be a binary relation. We say that an element $x \in \overline{R}$ is $R$-invariant [15] when $(x,x) \in \overline{R}$. We say that $R$ is concrete when every element in $\overline{R}$ is invariant, i.e., when $R$ is reflexive.

Let $\text{RRel} \to \text{ERel}$ be the full subcategory consisting of the concrete/reflexive relations. This embedding has both adjoints. The left adjoint $C : \text{ERel} \to \text{RRel}$ simply adds the diagonal:

$$CR := \{ \overline{R}, \overline{R} \cup \{(x,x)|x \in \overline{R}\} \}$$

The right adjoint $H : \text{ERel} \to \text{RRel}$ restricts the relation to its invariant elements, i.e., its reflexive centre:

$$HR := \{(x \in \overline{R}|x \in \overline{R}), \overline{R} \cap (HR)^2 \}$$

We summarise this situation in a diagram (in CAT):

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Using the coreflection $J \to H$, the following becomes a cartesian closed structure on $\text{RRel}$ [3, Proposition 27.9]:

$$R \times S := H(JR \times JS) \quad S^R := H((JS)^{JR})$$

Fig. 1 compares exponentials in ERel and RRel, where in general $\text{RRel}(R, S) \subseteq \text{Set}(\overline{R}, \overline{S})$.

The OHR model generalise this situation in two ways. The first one is to move to Kripke relations of varying arity, and the second is to move to $\omega$-chain-closed relations.

Example 2 (Kripke relations of varying arity). Fix a cardinal $\kappa$ bounding the arity of the relations. For finitary relations, we use the countable cardinal $\kappa := \aleph_0$. Let $\text{Set}_\kappa$ be the (small) full subcategory of Set consisting of the hereditarily $\kappa$-small sets. For each subcategory $\mathcal{C} \subseteq \text{Set}_\kappa$, consider the presheaf category $\mathcal{C} := [\mathcal{C}^{op}, \text{Set}]$. From the general theory of fibrations, $\text{Sub} \mathcal{C}$ has a bi-cartesian closed structure that is strictly preserved by the subobject fibration cod : $\text{Sub} \mathcal{C} \to \mathcal{C}$ [9, e.g.], as on the right:

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Taking the (small) product in CAT, ranging over $\mathcal{C}$, we have a bifibration with the same properties (to the left of cod), for which we can take the change-of-base along the functor sending each $X$ to the
This situation generalises binary relations: choosing \( \{ \text{one-object subcategory consisting of } f \} \) for logical relations by taking the direct image of the unit \([4]\). In Example 3, we have a forgetful functor \( R \rightarrow C \) and a right adjoint \( \Delta \), adding the diagonal to each relation, as well as a right adjoint \( \tilde{R} \). From the general theory of fibrations \([9]\), \( \tilde{R} \) respects the relations, i.e.:

\[
\forall \nu, \omega \in C \colon \{ \nu \} \subseteq \tilde{R}(\omega) \quad \text{if} \quad \nu \subseteq \omega \in C.
\]

Example 5 (free lifting). Taking the smallest lifting that is both compatible with the image and contains the least element \( \bot \), i.e., making each lifting \( M \) an admissible subset of \( L \). This is a special case of the free lifting \([7]\). We use this lifting.

The Eilenberg-Moore category for \( T \) consists of the admissible Kripke relations of varying arity, which is the crux of the OHR construction. To complete it, we note that in order to interpret a call-by-name language with the natural numbers as base type, we need an appropriate \( T \)-algebra. OHR bakes this choice of algebra into their category, but we want to separate it into the model structure.

4. Definability

Let \( \tau \) range over PCF types. In the final step in the construction, we use Katsumata’s \([8]\) definability characterisation using \( \tau \)-lifting. The \( \tau \)-lifting of \( L \) to \( \tilde{K} \) characterises the elements (approximated by definable elements). As the free lifting is contained in any lifting containing \( \bot \), and contains all the definable elements, we deduce that every element in \( \tilde{T} \) is approximated by definable elements, giving us the usual full-abstraction result. To use the \( \tau \)-lifting, we need to choose the cardinal \( \kappa \) to be large enough so that each \( [\tau] \) is \( \kappa \)-small. However, for PCF, OHR replace the definable elements by Milner’s finite definable approximations method \([11]\).

5. Prospects

We would like to transport this account to languages with arbitrary effects. As a starting point, we will consider a model \( \mathcal{C} \) for the programming language at hand, requiring it to be \( \omega \text{-Cpo} \)-enriched. We will then use the sub-scone \([12]\) to reconstruct a generalisation:

\[
\begin{array}{ccc}
\omega K & \xrightarrow{\bot} & \omega \mathcal{C} \\
\xrightarrow{\tilde{R}} & & \xrightarrow{\tilde{C}} \\
\omega \text{Sub} \tilde{\mathcal{C}} & \xrightarrow{\text{cod}} & \prod_{\mathcal{C} \in \text{Set}_\kappa} \mathcal{C}
\end{array}
\]

further assuming \( \mathcal{C} \) is sufficiently complete well-behaved for the adjoints to exist. As all of the constructions we have used, including the free lifting, and definability via \( \tau \)-lifting, are valid in this situation, we hope to break new ground in this general setting.

Acknowledgments

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Selected References


