Bifibrational functorial semantics of parametric polymorphism

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Abstract

Reynolds' theory of parametric polymorphism captures the invariance of polymorphically typed programs under change of data representation. Semantically, reflexive graph categories and fibrations are both known to give a categorical understanding of parametric polymorphism. This paper contributes further to this categorical perspective by showing the relevance of bifibrations. We develop a bifibrational framework for models of System F that are *parametric*, in that they verify the Identity Extension Lemma and Reynolds' Abstraction Theorem. We also prove that our models satisfy expected properties, such as the existence of initial algebras and final coalgebras, and that parametricity implies dinaturality.

Keywords: Parametricity, logical relations, System F, fibred category theory.

1 Introduction

Strachey [31] called a polymorphic function *parametric* if its behaviour is uniform across all of its type instantiations. Reynolds [26] made this mathematically precise by formulating the notion of *relational parametricity*, in which the uniformity of parametric polymorphic functions is captured by requiring them to preserve all logical relations between instantiated types. Relational parametricity has proven to be a key technique for formally establishing properties of software systems, such as representation independence [1,6], equivalences between programs [16], and useful ("free") theorems about programs from their types alone [32]. In this paper, we treat relational parametricity for the polymorphic λ-calculus System F [10], which forms the core of many modern programming languages and verification systems. Hermida, Reddy, and Robinson [15] give a good introduction to relational parametricity.

Since category theory underpins and informs many of the key ideas underlying modern programming languages, it is natural to ask whether it can provide a useful perspective on parametricity as well. Ma and Reynolds [20] developed the first categorical formulation of relational parametricity, but their models were complicated

This paper is electronically published in
Electronic Notes in Theoretical Computer Science
URL: `www.elsevier.nl/locate/entcs`

and challenging to understand. Moreover, Birkedal and Rosolini discovered that not all expected consequences of parametricity necessarily hold in their models (see [4]).

Another line of work, begun by O'Hearn and Tennent [22] and Robinson and Rosolini [29], and later refined by Dunphy and Reddy [7], uses reflexive graphs to model relations and functors between reflexive graph categories to model types. This is the state of the art for functorial semantics for parametric polymorphism. Interpreting types as functors is conceptually elegant and Dunphy and Reddy show that this framework is powerful enough to prove expected results, such as the existence of initial algebras for strictly positive type expressions [5]. However, since reflexive graph categories are relatively unknown mathematical structures, much of this development has had to be carried out from scratch. We propose to instead take the more established fibrational view of logic from the outset, and thus to analyse parametricity through the powerful lens of categorical type theory [17].

In doing so, we follow an extensive line of work by Hermida [12,13] and Birkedal and Møgelberg [4], who use fibrations to construct sophisticated categorical models not only of parametricity, but also of its logical structure in terms of Abadi-Plotkin logic [25]. Abadi-Plotkin logic is a formal logic for parametric polymorphism that includes predicate logic *and* a polymorphic lambda calculus, and thus requires significant machinery to handle. Using this machinery, Birkedal and Møgelberg are able to go beyond Dunphy and Reddy's results and, for instance, prove that all positive type expressions — not just the strictly positive ones as for Dunphy and Reddy — have initial algebras. However, these impressive results come at the price of the complexity of the notions involved. Our aim is to achieve the same results in a simpler setting [1], closer to Dunphy and Reddy's functorial semantics. We end up with a notion of model in which each type is interpreted as an equality preserving fibred functor and each term is interpreted as a fibred natural transformation. This is quite similar to the models produced by the parametric completion process of Robinson and Rosolini [29] (see also Birkedal and Møgelberg [4, Section 8]) and to Mitchell and Scedrov's relator model [21], but with a more general notion of relation given by a fibration. We thus combine the generality of Birkedal and Møgelberg's fibrational models with the simplicity of Dunphy and Reddy's functorial semantics.

Our central innovation is the use of *bifibrations* to achieve this "sweet spot" in the study of parametricity. This is not necessary for the *definition* of our framework, for which Lawvere equality [18] (i.e., opreindexing along diagonals only) suffices, but it helps considerably with both the concrete interpretation of ∀-types [9] and the handling of graph relations. At a technical level, our strongest result is to use our simpler framework to recover all the expected consequences of parametricity that Birkedal and Møgelberg [4] prove using Abadi-Plotkin logic. In particular, we go beyond Dunphy and Reddy's result by deriving, this time with a functorial semantics, initial algebras for all positive type expressions, rather than just for strictly positive ones. Nevertheless, this paper is in no way intended as the final word on fibrational parametricity. Instead, we hope the simple re-conceptualization of parametricity we offer here — replacing the usual categorical interpretations of types as functors and

[1] We stress again that we are not trying to model all of Abadi-Plotkin logic, but rather only type systems involving parametric polymorphism. Indeed, with respect to Abadi-Plotkin logic, we could not hope to improve upon the results of Birkedal and Møgelberg [4], who give a sound and complete semantics.

terms as natural transformations with their fibred counterparts — will open the way to the study of parametricity in richer settings, e.g., proof-relevant ones.

Structure of the paper: In Section 2 we give a short introduction to bifibrations. We recall Reynolds' relational interpretation of System F, the Identity Extension Lemma and the Abstraction Theorem in Section 3. We then extract bifibrational generalisations of these in Section 4, and construct our parametric models. In Section 5 we show that our models behave as expected by deriving initial algebras for all definable functors and proving that parametricity implies (di)naturality. Finally, we instantiate our framework to derive both "standard" and new models of relational parametricity in Section 6. Section 7 concludes and discusses future work.

2 A Fibrational Toolbox for Relational Parametricity

We give a brief introduction to fibrations; more details can be found in, e.g., [17].

Definition 2.1 Let $U : \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $g : Q \to P$ in $\mathcal{E}$ is *cartesian* over $f : X \to Y$ in $\mathcal{B}$ if $Ug = f$ and, for every $g' : Q' \to P$ in $\mathcal{E}$ with $Ug' = f \circ v$ for some $v : UQ' \to X$, there exists a unique $h : Q' \to Q$ with $Uh = v$ and $g' = g \circ h$. A morphism $g : P \to Q$ in $\mathcal{E}$ is *opcartesian* over $f : X \to Y$ in $\mathcal{B}$ if $Ug = f$ and, for every $g' : P \to Q'$ in $\mathcal{E}$ with $Ug' = v \circ f$ for some $v : Y \to UQ'$, there exists a unique $h : Q \to Q'$ with $Uh = v$ and $g' = h \circ g$.

We write $f^{\S}_P$ for the cartesian morphism over f with codomain P, and $f^P_{\S}$ for the opcartesian morphism over f with domain P. Such morphisms are unique up to isomorphism. If P is an object of $\mathcal{E}$ then we write f^*P for the domain of $f^{\S}_P$ and $\Sigma_f P$ for the codomain of $f^P_{\S}$.

Definition 2.2 A functor $U : \mathcal{E} \to \mathcal{B}$ is a *fibration* if for every object P of $\mathcal{E}$ and every morphism $f : X \to UP$ in $\mathcal{B}$, there is a cartesian morphism $f^{\S}_P : Q \to P$ in $\mathcal{E}$ over f. Similarly, U is an *opfibration* if for every object P of $\mathcal{E}$ and every morphism $f : UP \to Y$ in $\mathcal{B}$, there is an opcartesian morphism $f^P_{\S} : P \to Q$ in $\mathcal{E}$ over f. A functor U is a *bifibration* if it is both a fibration and an opfibration.

If $U : \mathcal{E} \to \mathcal{B}$ is a fibration, opfibration, or bifibration, then $\mathcal{E}$ is its *total category* and $\mathcal{B}$ is its *base category*. An object P in $\mathcal{E}$ is *over* its image UP and similarly for morphisms. A morphism is *vertical* if it is over id. We write $\mathcal{E}_X$ for the *fibre over* an object X in $\mathcal{B}$, i.e., the subcategory of $\mathcal{E}$ of objects over X and morphisms over id_X. For $f : X \to Y$ in $\mathcal{B}$, the function mapping each object P of $\mathcal{E}$ to f^*P extends to a functor $f^* : \mathcal{E}_Y \to \mathcal{E}_X$ mapping each morphism $k : P \to P'$ in $\mathcal{E}_Y$ to the morphism f^*k with $kf^{\S}_P = f^{\S}_{P'} f^*k$. The universal property of $f^{\S}_{P'}$ ensures the existence and uniqueness of f^*k. We call f^* the *reindexing functor along* f. A similar situation holds for opfibrations; the functor $\Sigma_f : \mathcal{E}_X \to \mathcal{E}_Y$ extending the function mapping each object P of $\mathcal{E}$ to $\Sigma_f P$ is the *opreindexing functor along* f.

We write $|\mathcal{C}|$ for the discrete category of $\mathcal{C}$. If $U : \mathcal{E} \to \mathcal{B}$ is a functor, then the *discrete functor* $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is induced by the restriction of U to $|\mathcal{E}|$. If $n \in \mathbb{N}$, then $\mathcal{E}^n$ denotes the n-fold product of $\mathcal{E}$ in Cat. The *n-fold product of* U, denoted $U^n : \mathcal{E}^n \to \mathcal{B}^n$, is the functor defined by $U^n(X_1, ..., X_n) = (UX_1, ..., UX_n)$.

Lemma 2.3 *If $U : \mathcal{E} \to \mathcal{B}$ is a functor then $|U| : |\mathcal{E}| \to |\mathcal{B}|$ is a bifibration. If U is a (bi)fibration then so is $U^n : \mathcal{E}^n \to \mathcal{B}^n$ for any natural number n.* □

To formulate Reynolds' relational parametricity categorically, we define the category Rel of relations over Set and the relations fibration on Set [17].

Definition 2.4 The category Rel has triples (A, B, R) as objects, where A, B, and R are sets and $R \subseteq A \times B$. A morphism $(A, B, R) \to (A', B', R')$ is a pair (f, g), where $f : A \to A'$ and $g : B \to B'$, such that if $(a, b) \in R$ then $(fa, gb) \in R'$.

We write (A, B, R) as just R when A and B are immaterial or clear from context. Note that Rel is *not* the category whose objects are sets and whose morphisms are relations, which also sometimes appears in the literature. Each set A has an associated *equality relation* defined by $\mathsf{Eq}\, A = \{(a, a) \mid a \in A\}$.

Example 2.5 The functor $U : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$ sending (A, B, R) to (A, B) is called the *relations fibration on* Set. To see that U is indeed a fibration, let $(f, g) : (X_1, X_2) \to (Y_1, Y_2)$ be a morphism in $\mathsf{Set} \times \mathsf{Set}$ with $UR = (Y_1, Y_2)$ for some R in Rel. If we define $(f, g)^* R \subseteq X_1 \times X_2$ by $(x_1, x_2) \in (f, g)^* R$ iff $(fx_1, gx_2) \in R$, then (f, g) is a cartesian morphism from $(f, g)^* R$ to R over (f, g). It is also easy to see that U is an opfibration, with opreindexing given by forward image. Thus, U is a bifibration. We denote the fibre over (A, B) in the relations fibration on Set by $\mathsf{Rel}(A, B)$.

Definition 2.6 Let $U : \mathcal{E} \to \mathcal{B}$ and $U' : \mathcal{E}' \to \mathcal{B}'$ be fibrations. A *fibred functor* $F : U \to U'$ comprises two functors $F_0 : \mathcal{B} \to \mathcal{B}'$ and $F_1 : \mathcal{E} \to \mathcal{E}'$ such that $U' F_1 = F_0 U$ and cartesian morphisms are preserved, i.e., if f is cartesian in $\mathcal{E}$ over g in $\mathcal{B}$ then $F_1 f$ is cartesian in $\mathcal{E}'$ over $F_0 g$ in $\mathcal{B}'$. If $F' : U \to U'$ is another fibred functor, then a *fibred natural transformation* $\eta : F \to F'$ comprises two natural transformations $\eta_0 : F_0 \to F_0'$ and $\eta_1 : F_1 \to F_1'$ such that $U' \eta_1 = \eta_0 U$.

In this paper we use fibred functors and fibred transformations to interpret System F types and terms, and show that under mild conditions this gives parametric models.

3 Reynolds' Model of Relational Parametricity

We now describe Reynolds' set-theoretic model of relational parametricity: first concretely, and then in terms of the relations fibration $\mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$. As Reynolds discovered, there are in fact no set-theoretic models if the meta-theory is classical logic [27], but the following makes sense in the (intuitionistic) internal language of a topos [23], or in the Calculus of Constructions with impredicative Set. Throughout, we assume a standard syntax for System F, given in Appendix A.1 for completeness.

3.1 Semantics of Types

Reynolds presents two "parallel" semantics for System F: a standard set-based semantics $[\![\_]\!]_o$, and a relational semantics $[\![\_]\!]_r$. Given $\Gamma \vdash T$ type, where the type context Γ contains $|\Gamma| = n$ type variables, Reynolds defines interpretations $[\![T]\!]_o : |\mathsf{Set}|^n \to \mathsf{Set}$ and $[\![T]\!]_r : |\mathsf{Rel}|^n(A, B) \to \mathsf{Rel}([\![T]\!]_o A, [\![T]\!]_o B)$ by structural induction on type judgements as follows:

- Type variables: $[\![X_i]\!]_o A = A_i$ and $[\![X_i]\!]_r R = R_i$
- Arrow types:

$$
\begin{aligned}
&[\![T_1 \to T_2]\!]_o A = [\![T_1]\!]_o A \to [\![T_2]\!]_o A \\
&[\![T_1 \to T_2]\!]_r R = \{(f, g) \mid (a, b) \in [\![T_1]\!]_r R \Rightarrow (fa, gb) \in [\![T_2]\!]_r R\}
\end{aligned}
$$

- Forall types:

$$
\begin{aligned}
[\![\forall X.T]\!]_o A \ &= \{f : \prod_{S:\mathsf{Set}} [\![T]\!]_o(A, S) \mid \forall R' \in \mathsf{Rel}(A', B') \,.\, (fA', fB') \in [\![T]\!]_r(\mathsf{Eq}\ A, R')\} \\
[\![\forall X.T]\!]_r R &= \{(f, g) \mid \forall R' \in \mathsf{Rel}(A', B') \,.\, (fA', gB') \in [\![T]\!]_r(R, R')\}
\end{aligned}
$$

The definitions of $[\![\forall X.T]\!]_o$ and $[\![\forall X.T]\!]_r$ depend crucially on one another. Thus, we do not really have two semantics — one based on Set and one based on Rel — but rather *a single semantics* based on the relations fibration $U : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$. In other words, Reynolds' definitions of $[\![-]\!]_o$ and $[\![-]\!]_r$ entail the following theorem:

Theorem 3.1 (Fibrational Semantics of Types) *Let U be the relations fibration on Set. Every judgement $\Gamma \vdash T$ induces a fibred functor $[\![T]\!] : |U|^{|\Gamma|} \to U$.*

$$
\begin{tikzcd}
|\mathsf{Rel}|^{|\Gamma|} \arrow[r, "{[\![T]\!]_r}"] \arrow[d, "|U|^{|\Gamma|}"'] & \mathsf{Rel} \arrow[d, "U"] \\
|\mathsf{Set}|^{|\Gamma|} \times |\mathsf{Set}|^{|\Gamma|} \arrow[r, "{[\![T]\!]_o \times [\![T]\!]_o}"'] & \mathsf{Set} \times \mathsf{Set}
\end{tikzcd}
$$

□

Since the domain of $[\![T]\!]_r$ is a discrete category, requiring that $[\![T]\!]$ is a fibred functor amounts simply to requiring that the above diagram commutes. In particular, no preservation of cartesian morphisms by $[\![T]\!]_r$ is needed. Reynolds does not give a functorial action of types on morphisms. This is reflected in the appearance of discrete categories in Theorem 3.1. As a result, Reynolds' pointwise interpretation of function spaces is the exponential in the functor category $|U|^{|\Gamma|} \to U$ [28]. How parametricity treats the action on morphisms will become clear in Section 5.1; instead of acting on morphisms, the interpretation of types act on *graph relations* induced by morphisms. For now, we simply note that the use of discrete domains does not take us out of the fibrational framework; Lemma 2.3 ensures that $[\![T]\!]$ is a functor between fibrations. The Identity Extension Lemma (IEL) is key for many applications of parametricity. It says that every relational interpretation preserves equality relations [2]:

Lemma 3.2 (IEL) *If $\Gamma \vdash T$ then $[\![T]\!]_r \circ |\mathsf{Eq}|^{|\Gamma|} = \mathsf{Eq} \circ [\![T]\!]_o$.* □

[2] Reynolds' approach also handles "identity relations" that are not equality relations, such as the information order on domains. In this paper, like many others [2,4,13,25], we only treat equality relations. In future work, we hope to give an axiomatic account of "identity relations" similar to that of Dunphy and Reddy [7].

3.2 Semantics of Terms

Reynolds' main result is his Abstraction Theorem, stating that all terms send related environments to related values. Reynolds first gives set-valued and relational interpretations of term contexts $\Delta = x_1 : T_1, \ldots, x_n : T_n$ by defining $\llbracket\Delta\rrbracket_o = \llbracket T_1\rrbracket_o \times \cdots \times \llbracket T_n\rrbracket_o$ and $\llbracket\Delta\rrbracket_r = \llbracket T_1\rrbracket_r \times \cdots \times \llbracket T_n\rrbracket_r$. This defines a fibred functor $\llbracket\Delta\rrbracket : |U|^{|\Gamma|} \to U$. Reynolds' then interprets each judgement $\Gamma; \Delta \vdash t : T$ as a family of functions $\llbracket t\rrbracket_o : \llbracket\Delta\rrbracket_o S \to \llbracket T\rrbracket_o S$ for each environment $S \in |\mathsf{Set}|^{|\Gamma|}$. We omit the standard definition of $\llbracket t\rrbracket_o$ here. Finally, Reynolds proves:

Theorem 3.3 (Abstraction Theorem) *Let $A, B \in \mathsf{Set}^{|\Gamma|}$, $R \in \mathsf{Rel}^{|\Gamma|}(A, B)$, $a \in \llbracket\Delta\rrbracket_o A$, and $b \in \llbracket\Delta\rrbracket_o B$. For every term $\Gamma; \Delta \vdash t : T$, if $(a, b) \in \llbracket\Delta\rrbracket_r R$, then $(\llbracket t\rrbracket_o\ A\ a, \llbracket t\rrbracket_o\ B\ b) \in \llbracket T\rrbracket_r R$. Or, more concisely, fibrationally: every judgement $\Gamma; \Delta \vdash t : T$ is interpreted as a fibred natural transformation $(\llbracket t\rrbracket_o, \llbracket t\rrbracket_r) : \llbracket\Delta\rrbracket \to \llbracket T\rrbracket$.*

$$\begin{tikzcd}
|\mathsf{Rel}|^{|\Gamma|} \arrow[r, bend left, "\llbracket\Delta\rrbracket_r"] \arrow[r, bend right, "\llbracket T\rrbracket_r"'] \arrow[r, phantom, "\Downarrow \llbracket t\rrbracket_r"] \arrow[d, "|U|^{|\Gamma|}"'] & \mathsf{Rel} \arrow[d, "U"] \\
|\mathsf{Set}|^{|\Gamma|} \times |\mathsf{Set}|^{|\Gamma|} \arrow[r, bend left, "\llbracket\Delta\rrbracket_o \times \llbracket\Delta\rrbracket_o"] \arrow[r, bend right, "\llbracket T\rrbracket_o \times \llbracket T\rrbracket_o"'] \arrow[r, phantom, "\Downarrow \llbracket t\rrbracket_o \times \llbracket t\rrbracket_o"] & \mathsf{Set} \times \mathsf{Set}
\end{tikzcd}$$

□

It is worthwhile to unpack the fibrational statement of the theorem: Since the domains of the functors $\llbracket\Delta\rrbracket_o$ and $\llbracket T\rrbracket_o$ are discrete, the interpretation $\llbracket t\rrbracket_o$ actually defines a (vacuously natural) transformation $\llbracket t\rrbracket_o : \llbracket\Delta\rrbracket_o \to \llbracket T\rrbracket_o$. By the definition of morphisms in the category Rel, the existence of the (again, vacuously natural) transformation $\llbracket t\rrbracket_r$ over $\llbracket t\rrbracket_o \times \llbracket t\rrbracket_o$ is exactly the statement that if $(a, b) \in \llbracket\Delta\rrbracket_r R$, then $(\llbracket t\rrbracket_o\, A\, a, \llbracket t\rrbracket_o\, B\, b) \in \llbracket T\rrbracket_r R$ — the verbose conclusion of the theorem.

Reynolds' original formulation of the Abstraction Theorem makes it seem at first glance as though it asserts a property of $\llbracket t\rrbracket_o$. Surprisingly, however, our fibrational version makes it clear that the Abstraction Theorem actually states the existence of additional algebraic structure given by $\llbracket t\rrbracket_r$ and, more generally, the interpretation of terms as fibred natural transformations. Taking this point of view and exposing this heretofore hidden stucture opens the way to our bifibrational generalisation of Reynolds' model.

4 Bifibrational Relational Parametricity

Thus far we have only shown how to view Reynolds' notion of parametricity in terms of the specific fibration $U : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$. We now generalise this to other fibrations. This requires that we generalise $\llbracket -\rrbracket_o$ and $\llbracket -\rrbracket_r$ in such a way that the IEL and the Abstraction Theorem hold, which in turn requires that we define equality functors for these other fibrations. The construction of equality functors is standard in any fibration with the necessary infrastructure [17], but we briefly describe it here for completeness. The first step is to note that the relations fibration from Example 2.5 arises from the subobject fibration over Set by so-called *change of base* (or pullback), and to generalise that construction.

Definition 4.1 Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration and suppose $\mathcal{B}$ has products. The fibration $\mathsf{Rel}(U) : \mathsf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ is defined by the following change of base:

$$\begin{tikzcd} \mathsf{Rel}(\mathcal{E}) \arrow[r, "q"] \arrow[d, "\mathsf{Rel}(U)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "U"] \\ \mathcal{B} \times \mathcal{B} \arrow[r, "\_\times\_"'] & \mathcal{B} \end{tikzcd}$$

We call $\mathsf{Rel}(U)$ the *relations fibration for* U, and call the objects of $\mathsf{Rel}(\mathcal{E})$ *relations* on $\mathcal{B}$, to emphasise that this construction generalises the relations fibration on Set. We say that a fibration $U : \mathcal{E} \to \mathcal{B}$ *has fibred terminal objects* if each fibre $\mathcal{E}_X$ of $\mathcal{E}$ has a terminal object, and if reindexing preserves these terminal objects. The map sending each object X of $\mathcal{B}$ to the terminal object in $\mathcal{E}_X$ extends to a functor $K : \mathcal{B} \to \mathcal{E}$ called the *truth functor* for U. We can construct an equality functor for $\mathsf{Rel}(U)$ from the truth functor for U as follows:

Definition 4.2 Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects. If $\mathcal{B}$ has products, then the map $X \mapsto \Sigma_{\delta_X} KX$, where δ_X is the diagonal morphism $\delta_X : X \to X \times X$, extends to the *equality functor* $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ for $\mathsf{Rel}(U)$.

For this definition, it is enough to ask for opreindexing along diagonals δ_X only (this is what Birkedal and Møgelberg [4] do to model equality). When dealing with graph relations in Section 5.1, though, we will use all of the opfibrational structure to opreindex along arbitrary morphisms. Our definition specialises to the equality relation $\mathsf{Eq}\,A$ when instantiated to the relations fibration on Set. The equality functor is faithful, but not always full; a counterexample is the equality functor for the identity bifibration $\mathsf{Id} : \mathsf{Set} \to \mathsf{Set}$, which gives a model with *ad hoc*, rather than parametric, polymorphic functions. We thus assume in the rest of this paper that equality functors are full. This is reminiscent of Birkedal and Møgelberg's [4] assumption that the fibration has *very strong equality*, i.e., that internal equality implies external equality, in the following sense: fullness says that if $(f, g, \alpha) : \mathbf{1} \to \mathsf{Eq}\,Y$ (i.e., α shows that $f = g$ internally), then, since $\mathbf{1} = \mathsf{Eq}\,\mathbf{1}_{\mathcal{B}}$, $(f, g, \alpha) = (h, h, \mathsf{Eq}\,h)$ for some $h : \mathbf{1}_{\mathcal{B}} \to Y$ (i.e., $f = g$ externally). We use fullness of Eq at several places in Section 5 below.

We now show how to interpret arrow types and forall types as fibred functors with discrete domains. We then show that a particular class of such functors forms a $\lambda 2$-fibration and thus a model of System F which is, in fact, parametric.

4.1 Interpreting Arrow Types

The definition of $[\![T \to U]\!]_o$ and $[\![T \to U]\!]_r$ in Section 3.1 is derived from the cartesian closed structure of Set and Rel, respectively. Moreover, the fibration $U : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$ preserves the cartesian closed structure, so that $[\![t]\!]_r$ is over $[\![t]\!]_o \times [\![t]\!]_o$ as required by the Abstraction Theorem. Generalising from this fibration, we can model arrow types "parametrically" — i.e., in a way satisfying the Abstraction Theorem — in any fibration $U : \mathcal{E} \to \mathcal{B}$ in which $\mathcal{E}$ and $\mathcal{B}$ are cartesian closed categories (CCCs) and U preserves cartesian closedness.

Definition 4.3 A fibration $U : \mathcal{E} \to \mathcal{B}$ is an *arrow fibration* if both $\mathcal{E}$ and $\mathcal{B}$ are

CCCs, and U preserves the cartesian closed structure. A relations fibration $\mathsf{Rel}(U)$ is an *equality preserving arrow fibration* if it is an arrow fibration and $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ preserves exponentials.

One advantage of working with well-studied mathematical structures such as fibrations is that many of their properties can be found in the literature. This helps in determining when a relations fibration is an equality preserving arrow fibration:

Lemma 4.4 *Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects and $\mathcal{B}$ be a CCC.*

(i) *If $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ has a left adjoint Q, then Eq preserves exponentials iff Q satisfies the Frobenius property. Such a Q exists if $U : \mathcal{E} \to \mathcal{B}$ has full comprehension, $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ is full and $\mathcal{B}$ has pushouts.*

(ii) *If $U : \mathcal{E} \to \mathcal{B}$ is a fibred CCC and has simple products (i.e., if, for every projection $\pi_B : A \times B \to A$ in $\mathcal{B}$, the reindexing functor π_B^* has a right adjoint and the Beck-Chevalley condition holds), then $\mathcal{E}$ is a CCC and U preserves the cartesian closed structure.* □

Change of base preserves simple products and fibred structure, so $\mathsf{Rel}(U)$ is a fibred CCC with simple products if U is. Moreover, $\mathcal{B} \times \mathcal{B}$ is a CCC if $\mathcal{B}$ is. Lemma 4.4 thus derives structure in $\mathsf{Rel}(U)$ from structure in U.

4.2 Interpreting Forall Types

We must generalise Reynolds' definitions of $[\![-]\!]_o$ and $[\![-]\!]_r$ for forall types to relations fibrations in such a way that the Abstraction Theorem and IEL hold. The rules for type abstraction and type application suggest that we should interpret $\forall$ as right adjoint to weakening by a type variable. We may first try to look for such an adjoint on the base category, then another on the total category, and then try to link these adjoints. But this is the wrong idea; for the relations fibration of Example 2.5, this gives *all* polymorphic functions, not just the parametrically polymorphic ones. Instead, we require an adjoint for the combined fibred semantics. Let $|\mathsf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ be the category whose objects are equality preserving fibred functors from $|\mathsf{Rel}(U)|^n$ to $\mathsf{Rel}(U)$ and whose morphisms are fibred natural transformations between them. Then:

Definition 4.5 $\mathsf{Rel}(U)$ is a $\forall$-*fibration* if, for every projection $\pi_n : |\mathsf{Rel}(U)|^{n+1} \to |\mathsf{Rel}(U)|^n$, the functor $\_ \circ \pi_n : (|\mathsf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathsf{Rel}(U)) \to (|\mathsf{Rel}(U)|^{n+1} \to_{\mathsf{Eq}} \mathsf{Rel}(U))$ has a right adjoint $\forall_n$ and this family of adjunctions is natural in n.

We write $\forall$ for $\forall_n$ when n can be inferred. This definition follows, e.g., Dunphy and Reddy [7] by "baking the Identity Extension Lemma into" the definition of forall types — in the sense that the very existence of $\forall$ requires that if F is equality preserving then so is $\forall F$ — rather than relegating it to a result to be proved *post facto*. If U is faithful, then Definition 4.5 can be reformulated in terms of more basic concepts using its opfibrational structure. The IEL then becomes a consequence of the definition, rather than an intrinsic part of it [9]. For the purposes of this paper, this abstract specification is enough.

4.3 Fibred functors with discrete domains form a parametric model

A $\lambda 2$-fibration, i.e., a fibration $p : \mathcal{G} \to \mathcal{S}$ with fibred finite products, finite products in $\mathcal{S}$, fibred exponents, a generic object Ω, and simple Ω-products, is a categorical model of System F. Seely [30] gives a sound interpretation of the calculus in such fibrations. We conclude this section with the following theorem:

Theorem 4.6 *If $\mathsf{Rel}(U)$ is an equality preserving arrow fibration and a $\forall$-fibration, then there is a $\lambda 2$-fibration in which types $\Gamma \vdash T$ are interpreted as equality preserving fibred functors $[\![T]\!] : |\mathsf{Rel}(U)|^{|\Gamma|} \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ and terms $\Gamma; \Delta \vdash t : T$ are interpreted as fibred natural transformations $[\![t]\!] : [\![\Delta]\!] \to [\![T]\!]$.* □

Note that Lemma 4.4 gives conditions for $\mathsf{Rel}(U)$ to be an arrow fibration, and our other paper [9] similarly gives conditions for $\mathsf{Rel}(U)$ to be a $\forall$-fibration. Unwinding the interpretation of System F in $\lambda 2$-fibrations [30], we see that we get the following for every fibration $U : \mathcal{E} \to \mathcal{B}$ satisfying the hypotheses of the theorem: for every System F type $\Gamma \vdash T$ and term $\Gamma; \Delta \vdash t : T$, we get

(i) a standard interpretation of $\Gamma \vdash T$ as a functor $[\![T]\!]_o : |\mathcal{B}|^{|\Gamma|} \to \mathcal{B}$;

(ii) a relational interpretation of $\Gamma \vdash T$ as a functor $[\![T]\!]_r : |\mathsf{Rel}(\mathcal{E})|^{|\Gamma|} \to \mathsf{Rel}(\mathcal{E})$;

(iii) a proof of the Identity Extension Lemma in the form of Lemma 3.2, i.e., a proof that $[\![T]\!]$ is equality preserving;

(iv) a standard interpretation of $\Gamma; \Delta \vdash t : T$ as a natural transformation $[\![t]\!]_o : [\![\Delta]\!]_o \to [\![T]\!]_o$; and

(v) a proof of the Abstraction Theorem in the form of Theorem 3.3, i.e., a proof that $\Gamma; \Delta \vdash t : T$ has a relational interpretation as a natural transformation $[\![t]\!]_r : [\![\Delta]\!]_r \to [\![T]\!]_r$ over $[\![t]\!]_o \times [\![t]\!]_o$.

Theorem 4.6 also gives a powerful internal language [17], where base types in type context Γ are given by fibred functors $|\mathsf{Rel}(U)|^{|\Gamma|} \to_{\mathsf{Eq}} \mathsf{Rel}(U)$, and base term constants in term context Δ are given by fibred natural transformations $[\![\Delta]\!] \to [\![T]\!]$. Thus, we can use this language to reason about our models using System F. This will be used in the proofs of Theorems 5.7 and 5.11 below.

5 Consequences of parametricity

We use our new framework to derive expected consequences of parametricity. This serves as a "sanity check" for our new bifibrational conceptualisation, and shows that our framework is powerful enough to derive the same results as, e.g., Birkedal and Møgelberg [4]. At a high-level, our proof strategies are often similar to the ones found in the literature, while the proofs of individual facts are necessarily specific to our setting, and often fibrational in nature.

5.1 Graph Relations

In the fibration $U : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$ every function $f : X \to Y$ defines a graph relation $\langle f \rangle = \{(x, y) \mid fx = y\} \subseteq X \times Y$. This generalises to the fibrational setting, where the graph of $f : A \to B$ is obtained by reindexing the equality relation on B.

Definition 5.1 Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration with fibred terminal objects and products in $\mathcal{B}$. The *graph* of $h : X \to Y$ in $\mathcal{B}$ is $\langle h \rangle = (h, \mathsf{id}_Y)^*(\mathsf{Eq}\, Y)$ in $\mathsf{Rel}(\mathcal{E})$.

The definition of $\langle h \rangle$ agrees with the set-theoretic one for the relations fibration on Set. Since reindexing preserves identities, $\langle \mathsf{id}_A \rangle = (\mathsf{id}_A, \mathsf{id}_A)^*(\mathsf{Eq}\, A) = \mathsf{Eq}\ A$ for any object A of $\mathcal{B}$. In a bifibration, we can also define the graph of $f : A \to B$ in another, isomorphic way by using opfibrational structure to opreindex equality on A:

Lemma 5.2 (Lawvere [18]) *If $U : \mathcal{E} \to \mathcal{B}$ is a bifibration with fibred terminal objects that satisfies the Beck-Chevalley condition [17, Section 1.8.11], and if $\mathcal{B}$ has products, then the graph of $h : X \to Y$ can also be described by $\langle h \rangle = \Sigma_{(\mathsf{id}_X, h)}(\mathsf{Eq}\, X)$.*□

Being able to describe graph relations in terms of *either* reindexing *or* opreindexing in any bifibration lets us use the universal properties of both when proving theorems about them. Graph relations are the key structures that turn morphisms in $\mathcal{B}$ into objects in $\mathsf{Rel}(\mathcal{E})$ and, more generally, mediate the standard and relational semantics.

The *graph functor* for $\mathsf{Rel}(U) : \mathsf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ is the functor $\langle \_ \rangle : \mathcal{B}^{\to} \to \mathsf{Rel}(\mathcal{E})$ mapping $f : X \to Y$ in $\mathcal{B}$ to $\langle f \rangle$ in $\mathsf{Rel}(\mathcal{E})$. To see how $\langle \_ \rangle$ acts on morphisms, recall that if $f : X \to Y$ and $f' : X' \to Y'$ are objects of $\mathcal{B}^{\to}$, then a morphism from f to f' is a pair of morphisms $g : X \to X'$ and $h : Y \to Y'$ such that $f' \circ g = h \circ f$. The universal property of reindexing in $\mathsf{Rel}(U)$ guarantees the existence of a unique morphism $\langle g, h \rangle : \langle f \rangle \to \langle f' \rangle$ over (g, h) such that the following diagram commutes:

$$
\begin{tikzcd}
\langle f \rangle \arrow[r, "(f{,}\mathsf{id})^{\S}"] \arrow[d, dashed, "\exists ! \langle g{,}h \rangle"'] & \mathsf{Eq}\ Y \arrow[d, "\mathsf{Eq}\ h"] \\
\langle f' \rangle \arrow[r, "(f'{,}\mathsf{id})^{\S}"'] & \mathsf{Eq}\ Y'
\end{tikzcd}
$$

Lemma 5.3 *If the underlying bifibration satisfies the Beck-Chevalley condition, then $\langle \_ \rangle : \mathcal{B}^{\to} \to \mathsf{Rel}(\mathcal{E})$ is full and faithful if $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ is.* □

The proof uses the opfibrational characterisation of the graph functor from Lemma 5.2. The main tool for deriving consequences of parametricity is the Graph Lemma, which relates the graph of the action of a functor on a morphism with its relational action on the graph of the morphism. Interestingly, although our setting is possibly proof-relevant (i.e., there can be multiple proofs that two elements are related), the following "logical equivalence" version of the Graph Lemma is strong enough for our applications. If $U : \mathcal{E} \to \mathcal{B}$ and $U' : \mathcal{E}' \to \mathcal{B}'$ are fibrations, we write $(F_o, F_r) : \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U')$ to indicate that functors (not necessarily fibred) $F_o : \mathcal{B} \to \mathcal{B}'$ and $F_r : \mathsf{Rel}(\mathcal{E}) \to \mathsf{Rel}(\mathcal{E}')$ are such that $\mathsf{Rel}(U') \circ F_r = (F_o \times F_o) \circ \mathsf{Rel}(U)$, and (F_o, F_r) is equality preserving, i.e., $F_r \circ \mathsf{Eq} = \mathsf{Eq} \circ F_o$.

Theorem 5.4 (Graph Lemma) *Assume the underlying bifibration satisfies the Beck-Chevalley condition, and let $(F_o, F_r) : \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U)$. For any $h : X \to Y$ in $\mathcal{B}$, there are vertical morphisms $\phi_h : \langle F_o h \rangle \to F_r \langle h \rangle$ and $\psi_h : F_r \langle h \rangle \to \langle F_o h \rangle$ in $\mathsf{Rel}(\mathcal{E})$.* □

Our proof of the Graph Lemma is completely independent of the specific functor (F_o, F_r), and so in particular does not proceed by induction on the structure of types. This is a key reason why we can go beyond Dunphy and Reddy [7] and prove the existence of initial algebras of positive, rather than just strictly positive, type expressions.

5.2 Existence of Initial Algebras

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. An *F-algebra* is a pair (A, k_A) with A an object of $\mathcal{C}$ and $k_A : FA \to A$ a morphism. We call A the *carrier* of the F-algebra and k_A its *structure map*. A morphism $h : A \to B$ in $\mathcal{C}$ is an *F-algebra homomorphism* $h : (A, k_A) \to (B, k_B)$ if $k_B \circ (Fh) = h \circ k_A$. An F-algebra (Z, in) is *weakly initial* if, for any F-algebra (A, k_A), there exists a mediating F-algebra homomorphism $\mathit{fold}[A, k_A] : (Z, in) \to (A, k_A)$. It is an *initial F-algebra* if $\mathit{fold}[A, k_A]$ is unique.

The literature contains other proofs that initial algebras exist in parametric models (e.g., [4,25]). Closest to our setting is Dunphy and Reddy [7], who show that strictly positive types have initial algebras. Under assumptions no stronger than theirs, we sharpen this result to *all* positive types, or, more generally, all functors on our parametric models that are strong (see below) and equality preserving.

Let $F = (F_o, F_r) : \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ be a functor (note that the domain of F is not discrete and that F need not preserve cartesian morphisms) with a *strength* $t = (t_o, t_r)$, i.e., a family of morphisms $(t_o)_{A,B} : A \Rightarrow B \to F_o A \Rightarrow F_o B$ and $(t_r)_{R,S} : R \Rightarrow S \to F_r R \Rightarrow F_r S$ with $(t_r)_{R,S}$ over $((t_o)_{A,B}, (t_o)_{C,D})$ if R is over (A, B) and S is over (C, D), such that t preserves identity and composition. A functor with a strength is said to be *strong*. Because of the discrete domains, t is a natural transformation from $\_ \Rightarrow \_$ to $F\_ \Rightarrow F\_$ in $|\mathsf{Rel}(U)|^2 \to_{\mathsf{Eq}} \mathsf{Rel}(U)$, and thus $\alpha, \beta; \cdot \vdash \underline{t} : (\alpha \to \beta) \to (\underline{F}[\alpha] \to \underline{F}[\beta])$ represents the action of F on morphisms in the internal language (See Appendix A.2 for more details). All type expressions with one free type variable occurring only positively give rise to strong functors, but there are further examples of such functors, for instance if the model contains non-System F type constructions with natural functorial (and relational) interpretations — for example, those of dependent types in Set. We will show that an initial F_o-algebra exists. For this, we first construct a weak initial F_o-algebra, which can be done in any $\lambda 2$-fibration. Using the internal language, we define Z by $(Z_o, Z_r) = [\![\forall X.(\underline{F} X \to X) \to X]\!]$.

Lemma 5.5 *Z_o is the carrier of a weak initial F_o-algebra (Z_o, in_o) with mediating morphism $\mathit{fold}_o[A, k]$ and Z_r is the carrier of a weak initial F_r-algebra (Z_r, in_r) with mediating morphism $\mathit{fold}_r[A, k]$.* □

To show that fold_o is unique, we use the graph relations from Section 5.1. Recall that a category with a terminal object $\mathbf{1}$ is *well-pointed* if, for any $f, g : A \to B$, we have $f = g$ iff $f \circ e = g \circ e$ for all $e : \mathbf{1} \to A$. Like Dunphy and Reddy [7], we only consider well-pointed base categories; well-pointedness is used to convert internal language reasoning in non-empty contexts to closed contexts, so that we can apply semantic techniques such as Theorem 5.4.

Lemma 5.6 *Assume that the underlying bifibration satisfies the Beck-Chevalley condition, and that Eq is full.*

(i) *If $\mathcal{B}$ is well-pointed, then $\mathit{fold}_o[Z_o, in_o] = \mathsf{id}_Z$.*

(ii) *For every F_o-algebra homomorphism $h : (Z_o, in_o) \to (A, k_A)$, we have that $h \circ fold_o[Z_o, in_o] = fold_o[A, k_A]$.* □

The proofs of the two parts of Lemma 5.6 are similar: both use the graph functor to map commuting diagrams in $\mathcal{B}$ to morphisms in $\mathsf{Rel}(\mathcal{E})$, and then use the Graph Lemma to see that these morphisms are F_r-algebras. Lemma 5.5 and Lemma 5.6 together now immediately imply the main result.

Theorem 5.7 *If the underlying bifibration satisfies the Beck-Chevalley condition,* Eq *is full, and $\mathcal{B}$ is well-pointed, then (Z_o, in_o) is an initial F_o-algebra.* □

We show in Section 6 that these hypothesis cannot be weakened. One may wonder if the above result can be strengthened to get not only an initial F_o-algebra, but also an initial F_r-algebra. Certainly this is possible for the relations fibration $\mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$, since relations in Rel are proof irrelevant: maps either preserve relatedness or not. This translates in the axiomatic bifibrational setting to requiring the fibration $\mathsf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ to be faithful. When it is, the weak initial F_r-algebra is, in fact, initial: faithfulness implies the required uniqueness.

5.3 Existence of final coalgebras

We can also dualise the proof from the previous section to show the existence of final coalgebras in the usual manner [11]. As usual, this requires us to first encode products and existential types in System F. We encode products as $A \times B = \forall Y.(A \to B \to Y) \to Y$. This supports the usual pairing and projection operations, as well as surjective pairing using parametricity. We encode existential types by $\exists X.T = \forall Y.(\forall X.(T \to Y)) \to Y$. We can support introduction and elimination rules

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma; \Delta \vdash u : T[A/X]}{\Gamma; \Delta \vdash \langle A, u \rangle : \exists X.T(X)} \qquad \frac{\Gamma; \Delta \vdash t : \exists X.T \quad \Gamma, Z; \Delta, y : T[Z/X] \vdash s : S}{\Gamma; \Delta \vdash (\mathsf{open}\, t\, \mathsf{as}\, \langle Z, y \rangle\, \mathsf{in}\, s) : S}$$

with the conversion $\mathsf{open}\,\langle A, t \rangle\,\mathsf{as}\,\langle Z, y \rangle\,\mathsf{in}\, s = s[X/A, y/t]$ by defining $\langle A, t \rangle = \Lambda Y.\lambda f.f\, A\, t$ and $\mathsf{open}\, t\,\mathsf{as}\,\langle Z, y \rangle\,\mathsf{in}\, s = t\, V\, (\Lambda Z.\lambda y.s)$. Using parametricity we can prove the following commutation property and η-rule for existential types:

Lemma 5.8 *Assume the underlying bifibration satisfies the Beck-Chevalley condition, and that* Eq *is full.*

(i) *Let $\Gamma; \Delta \vdash t : \exists X.T$, let $\Gamma, Z; \Delta, u : T[Z/X] \vdash s : S$ and let $\Gamma; \Delta \vdash f : S \to S'$ for a closed type S'. Then $[\![f(\mathsf{open}\, t\, \mathsf{as}\, \langle Z, u \rangle\, \mathsf{in}\, s)]\!]_o = [\![\mathsf{open}\, t\, \mathsf{as}\, \langle Z, u \rangle\, \mathsf{in}\, f(s)]\!]_o$.*

(ii) *If $\Delta; \Gamma \vdash t : \exists X.T$, then $[\![\mathsf{open}\, t\, \mathsf{as}\, \langle Z, u \rangle\, \mathsf{in}\, \langle Z, u \rangle]\!]_o = [\![t]\!]_o$.* □

If $F : \mathcal{C} \to \mathcal{C}$ is an endofunctor, an *F-coalgebra* is a pair (A, k_A) with A an object of $\mathcal{C}$ and $k_A : A \to FA$ a morphism. We call A the *carrier* of the F-coalgebra and k_A its *structure map*. A morphism $h : A \to B$ in $\mathcal{C}$ is an *F-coalgebra homomorphism* $h : (A, k_A) \to (B, k_B)$ if $k_B \circ h = Fh \circ k_A$. An F-coalgebra (W, out) is *weakly final* if, for any F-coalgebra (A, k_A), there exists a mediating F-coalgebra homomorphism $unfold[A, k_A] : (A, k_A) \to (W, out)$. It is a *final F-coalgebra* if $unfold[A, k_A]$ is unique.

Let $F = (F_o, F_r) \colon \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ be a functor with a strength t. We show that the final F_o-coalgebra exists. Again, we first construct a weakly final coalgebra by defining $W = (W_o, W_r) = [\![\exists X.(X \to \underline{F}(X)) \times X]\!]$.

Lemma 5.9 *W_o is the carrier of a weakly final F_o-coalgebra (W_o, out_o) with mediating morphism $unfold_o[A, k]$ and W_r is the carrier of a weakly final F_r-coalgebra (W_r, out_r) with mediating morphism $unfold_r[A, k]$.* □

We proceed similarly to Lemma 5.6. This time, we use the opfibrational part of the Graph Lemma to construct F_r-coalgebras.

Lemma 5.10 *Assume the underlying bifibration satisfies the Beck-Chevalley condition, and that* Eq *is full.*

(i) *For every F_o-coalgebra morphism $h : (A, k_A) \to (B, k_B)$ we have $unfold_o[B, k_B] \circ h = unfold_o[A, k_A]$.*

(ii) $unfold_o[W_o, out_o] = \mathsf{id}_{W_o}$. □

Putting things together, we have constructed a final coalgebra.

Theorem 5.11 *If the underlying bifibration satisfies the Beck-Chevalley condition, and if* Eq *is full, then (W_o, out_o) is a final F_o-coalgebra.* □

5.4 Parametricity Implies Dinaturality

We show that our axiomatic foundations can be used to prove that dinaturality can be deduced from parametricity. This is well-known in other settings (see, e.g., [4, Section 5.1]), but we do it because i) it shows our foundation passes this test; and ii) it highlights again the use of bifibrations to give two definitions of the graph of a function both of which are used in the proof. First, the definition of dinaturality:

Definition 5.12 If $F, G : \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{B}$ are mixed variant functors, then a dinatural transformation $t : F \to G$ is a collection of morphisms $t_X : FXX \to GXX$ indexed by objects X of $\mathcal{B}$ such that, for every $g : X \to Y$ of $\mathcal{B}$, the following commutes:

$$\begin{tikzcd}
 & F\,X\,X \arrow[r, "t_X"] & G\,X\,X \arrow[dr, "G(\mathsf{id}_X{,}g)"] & \\
F\,Y\,X \arrow[ur, "F(g{,}\mathsf{id}_X)"] \arrow[dr, "F(\mathsf{id}_Y{,}g)"'] & & & G\,X\,Y \\
 & F\,Y\,Y \arrow[r, "t_Y"'] & G\,Y\,Y \arrow[ur, "G(g{,}\mathsf{id}_Y)"'] &
\end{tikzcd}$$

We note that our proof applies to all mixed variant functors with equality preserving liftings, not just strong such functors.

Theorem 5.13 *Let $(F_o, F_r), (G_o, G_r) : \mathsf{Rel}(U)^{op} \times \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U)$. Further, let $t^0_A : F_o AA \to G_o AA$ be a family indexed by objects A of $\mathcal{B}$, and $t^1_R : F_r RR \to G_r RR$ be a family indexed by objects R of $\mathsf{Rel}(\mathcal{E})$ such that if R is over (A, B), then t^1_R is over $(t^0 A, t^0 B)$. Then t^0 is a dinatural transformation from F_o to G_o.* □

Theorem 5.13 applies in particular to the interpretation of terms $t : \forall X.FXX \to GXX$ where F and G are given by type expressions with two free type variables, one occuring positively and one negatively. As is well known, dinaturality reduces to naturality when F and G are covariant.

6 Examples

The construction of examples remains delicate — for instance, there are no set-theoretic models with a classical meta-theory. We give five models: Examples 6.1, 6.3, 6.4 and 6.5 are to be regarded as being internal to the Calculus of Constructions with impredicative Set (with $\neg\neg$-stable equality for Example 6.3), while Example 6.2 is internal to the category of ω-sets.

Before doing so, we take a moment to emphasise the generality of our framework. Considering different fibrations, we can derive parametric models with very different flavours. For example, changing the base category of the fibration corresponds to changing the 'standard' model in which we interpret types and terms. Changing the total category and the fibration (i.e., the functor itself) corresponds to changing the relevant notion of relational logic. We take advantage of the possibility of non-standard relations in Examples 6.2, 6.3 and Non-example 6.5.

Example 6.1 Reynolds' set-theoretic model is an instance of our framework via the relations fibration on Set. The equality functor is full and faithful in this bifibration, and Set is well-pointed. Hence Theorems 5.7 and 5.13 ensure that initial algebras exist, and that all terms are interpreted as dinatural transformations.

Example 6.2 The PER model of Bainbridge et al. [2] is an instance of our framework, if bifibrations are understood as internal to the category of ω-sets, so that natural transformations are *uniformly* realised (see also Longo and Moggi [19] for a detailed construction of a model using a category of PERs internal to ω-sets). An object of the category $\mathsf{PER}_{\mathbb{N}}$ is a symmetric, transitive relation $R \subseteq \mathbb{N} \times \mathbb{N}$. A morphism from R to S is a function $f : \mathbb{N}/R \to \mathbb{N}/S$ that is *tracked* by some partial recursive function $\phi_k : \mathbb{N} \rightharpoonup \mathbb{N}$, i.e., such that $f([n]_R) = [\phi_k(n)]_S$ for all $[n]_R \in \mathbb{N}/R$. The appropriate notion of predicate with respect to a PER R is that of a *saturated* subset, i.e., a subset $P \subseteq \mathbb{N}$ such that $P(x)$ and $R(x, x')$ implies $P(x')$. Saturated subsets form a bifibration over PERs with a full and faithful equality functor $\mathsf{Eq}A = A$. The CCC structure of $\mathsf{PER}_{\mathbb{N}}$ and SatRel is standard; a bijective pairing function $\langle \_, \_ \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ gives the product and recursion theory (the s-m-n Theorem) gives the exponential. The interesting case is that of forall types, which are interpreted as (cut-down, to ensure equality preservingness) intersections of PERs: $[\![\forall X.F]\!]_o \bar{R} = \{(n,k) \in \bigcap_{R':\mathsf{PER}_{\mathbb{N}}} [\![F]\!]_o(\bar{R}, R') |$ for all $Q : \mathsf{SatRel}(S,T).(n,n),(k,k) \in [\![F]\!]_r(\mathsf{Eq}(\bar{R}), Q)\}$ and $[\![\forall X.F]\!]_r \bar{P} = \bigcap_{Q:\mathsf{SatRel}(R,S)} [\![F]\!]_r(\bar{P}, Q)$. Since $\mathsf{PER}_{\mathbb{N}}$ is also well-pointed, Theorems 5.7 and 5.13 again apply.

Example 6.3 The previous models are well-known, but our framework also suggests new ones. A relation $R \subseteq X \times Y$ can be understood classically as a function from $X \times Y$ to Bool. (Constructively, this only covers *decidable* relations.) Here, Bool can be replaced with any constructively completely distributive [8] non-trivial lattice $\mathcal{V}$ of "truth values", leading to "multi-valued parametricity". For instance, the collection $\mathcal{D}(L)$ of all down-closed subsets of a complete lattice L is constructively completely distributive, and classically, we recover Bool as $\mathcal{D}(\mathbf{1})$. The category $\mathsf{Fam}(\mathcal{V})$ has objects (A, p), where A is a set and $p : A \to \mathcal{V}$ is thought of as a $\mathcal{V}$-valued predicate. The families fibration $\pi : \mathsf{Fam}(\mathcal{V}) \to \mathsf{Set}$ is a bifibration with $\Sigma_f(Q)(y) = \sup_{fx=y} Q(x)$, fibred terminal objects $(X, \lambda\,\_.\,\top)$, where $\top$ is the greatest element of $\mathcal{V}$, and comprehension given by $\{(A,p)\} = p^{-1}(\top)$. Since $\mathcal{V}$ is complete, it is a Heyting algebra, so that $\pi : \mathsf{Fam}(\mathcal{V}) \to \mathsf{Set}$ is a fibred CCC. Also, π has simple products given by $\Pi_\pi(A \times B, p)(a) = \inf_{x \in B}\ p(a,b)$. By Lemma 4.4, $\mathsf{Rel}(\pi)$ is thus an equality preserving arrow fibration. Finally, the interpretation of forall types is given by $[\![\forall X.F]\!]_o \bar{A} = \{f : \prod_{S:\mathsf{Set}} [\![F]\!]_o(\bar{A}, S) \mid \inf_{P:X\times Y \to \mathcal{V}} [\![F]\!]_r(\mathsf{Eq}\bar{A}, P) = \top\}$ and $[\![\forall X.F]\!]_r \bar{P} = \inf_{Q:X\times Y\to\mathcal{V}} [\![F]\!]_r(\bar{P}, Q)$. Distributivity is used to show that this functor is equality preserving. Fullness of Eq is obvious by $\neg\neg$-stable equality.

The extra conditions we impose in Section 5 really are necessary: the following are examples of $\forall$- and arrow-fibrations where Theorems 5.7 and 5.13 fail.

Non-example 6.4 Let G be a fixed (non-trivial) group, and consider the category of G-sets, i.e., the category with objects $(X, \cdot_X)$, where X is a set and $\cdot_X : G \times X \to X$ is a group action. Morphisms are functions between the carrier sets that respect the group action. Together with equivariant (i.e., group action respecting) relations, this forms a bifibration that is a model of System F in the sense of Theorem 4.6. However, the category is not well-pointed, and we can see that this makes Theorem 5.7 fail in our setting: the interpretation of $\forall X\,.\,X \to X$ is not the singleton G-set $\mathbf{1}$ as expected, but instead contains all the elements of the group G. We conjecture that this non-example also extends to a constructive treatment of the category of nominal sets [24].

Non-example 6.5 The identity fibration $\mathsf{Id} : \mathsf{Set} \to \mathsf{Set}$ models *ad hoc* polymorphism: it is a $\forall$- and arrow-fibration, but the equality functor $\mathsf{Eq}\,X = X \times X$ is not full. This explains why Theorem 5.13 fails: $[\![\forall X.T]\!]_o$ includes *ad hoc* polymorphic functions, so that e.g. $[\![\forall X.X \to X]\!]_o$ contains non-natural transformations such as η, where $\eta_{\mathsf{Bool}}(x) = \neg x$ and $\eta_X(x) = x$ for $X \neq \mathsf{Bool}$.

7 Conclusions and future work

Our interpretation of types and terms as fibred functors and fibred natural transformations shows that parametricity entails replacing the usual categorical semantics involving categories, functors, and transformations with one based on fibrations, fibred functors, and fibred transformations. The results in Section 5 show that our new approach based on *bifibrations* hits the sweet spot of a light structure that still suffices to prove key results. Work is ongoing in using the bifibrational framework to develop new notions such as proof-relevant parametricity, and higher order parametricity with interesting links to cubical sets that also appear in the semantics of Homotopy Type Theory [3].

Acknowledgement

We thank the reviewers of this and previous versions of the paper for their comments and suggestions. We especially thank Uday Reddy for extremely valuable advice and encouragement. Research supported by EPSRC grants EP/K023837/1 (NG, FNF), EP/M016951/1 (NG), NSF award 1420175 (PJ), and SICSA (FO).

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A System F: syntax and semantics

In this appendix we collect some standard facts about System F for the convenience of the reviewers.

A.1 The Syntax and Rules of System F

A *type context* is a list of type variables $X_1, ..., X_n$. A *type judgement* is of the form $\Gamma \vdash T$ type, where Γ is a type context. These judgements are defined inductively as follows:

$$\frac{X_i \in \Gamma}{\Gamma \vdash X_i \text{ type}} \qquad \frac{\Gamma \vdash T_1 \text{ type} \quad \Gamma \vdash T_2 \text{ type}}{\Gamma \vdash T_1 \to T_2 \text{ type}} \qquad \frac{\Gamma, X \vdash T \text{ type}}{\Gamma \vdash \forall X.T \text{ type}}$$

We consider α-convertible types equivalent. If desired, base types or other type constants $\Gamma \vdash C$ type can also be added to the language.

A *term context* is of the form $\Gamma \vdash \Delta$, where Γ is a type context, Δ is of the form $x_1 : T_1, ..., x_m : T_m$ for distinct x_i, and there is a type judgement $\Gamma \vdash T_i$ type for each $i \in \{1, ..., m\}$. A *term judgement* is of the form $\Gamma; \Delta \vdash t : T$, where Γ is a type context, $\Gamma \vdash \Delta$ is a term context, and $\Gamma \vdash T$ is a type judgement. The term judgements are defined inductively as follows:

$$\frac{x_i : T_i \in \Delta}{\Gamma; \Delta \vdash x_i : T_i} \qquad \frac{\Gamma; \Delta, x : T_1 \vdash t : T_2}{\Gamma; \Delta \vdash \lambda x.t : T_1 \to T_2} \qquad \frac{\Gamma; \Delta \vdash t_1 : T_1 \quad \Gamma; \Delta \vdash t_2 : T_1 \to T_2}{\Gamma; \Delta \vdash t_2\, t_1 : T_2}$$

$$\frac{\Gamma, X; \Delta \vdash t : T}{\Gamma; \Delta \vdash \Lambda X.t : \forall X.T} \qquad \frac{\Gamma; \Delta \vdash t : \forall X.T \quad \Gamma \vdash A \text{ Type}}{\Gamma; \Delta \vdash t\, A : T[X \mapsto A]}$$

Type abstraction requires that X not appear in Δ. Capture-free substitution of the type A for the free occurrences of X in the type T is denoted $T[X \mapsto A]$. As for types, term constants $\Gamma; \Delta \vdash c : T$ can be added if desired.

We have the following conversion rules for terms:

$$\frac{}{\Gamma; \Delta \vdash \lambda x.\, t = \lambda y.\, t[x \mapsto y] : T_1 \to T_2}\ (\alpha_\lambda) \qquad \frac{}{\Gamma; \Delta \vdash \Lambda X.\, t = \Lambda Y.\, t[X \mapsto Y] : \forall X.T}\ (\alpha_\Lambda)$$

$$\frac{}{\Gamma; \Delta \vdash (\lambda x.\, t)\, s = t[x \mapsto s] : T_2}\ (\beta_\lambda) \qquad \frac{}{\Gamma; \Delta \vdash (\Lambda X.\, t)[A] = t : T[X \mapsto A]}\ (\beta_\Lambda)$$

$$\frac{x \notin FV(t)}{\Gamma; \Delta \vdash t = \lambda x.\, t\, x : T_1 \to T_2}\ (\eta_\lambda) \qquad \frac{X \notin FTV(t)}{\Gamma; \Delta \vdash t = \Lambda X.\, t\, X : \forall X.T}\ (\eta_\Lambda)$$

$$\frac{\Gamma; \Delta, \vdash t_1 = t_2 : T_1 \to T_2 \qquad \Gamma; \Delta, \vdash s_1 = s_2 : T_1}{\Gamma; \Delta \vdash t_1\, s_1 = t_2\, s_2 : T_2}\ \mathrm{cong}_\lambda$$

$$\frac{\Gamma; \Delta, \vdash t_1 = t_2 : \forall X.T}{\Gamma; \Delta \vdash t_1\, A = t_2\, A : T[X \mapsto A]}\ \mathrm{cong}_\Lambda$$

$$\frac{\Gamma; \Delta, x : T_1 \vdash t_1 = t_2 : T_2}{\Gamma; \Delta \vdash \lambda x.\, t_1 = \lambda x.\, t_2 : T_1 \to T_2}\ (\xi_\lambda) \qquad \frac{\Gamma, X; \Delta \vdash t_1 = t_2 : T}{\Gamma; \Delta \vdash \Lambda X.\, t_1 = \Lambda X.\, t_2 : \forall X.T}\ (\xi_\Lambda)$$

$$\frac{}{\Gamma; \Delta \vdash t = t : T}\ \mathrm{refl} \qquad \frac{\Gamma; \Delta \vdash s = t : T}{\Gamma; \Delta \vdash t = s : T}\ \mathrm{sym}$$

$$\frac{\Gamma; \Delta \vdash t = s : T \qquad \Gamma; \Delta \vdash s = u : T}{\Gamma; \Delta \vdash t = u : T}\ \mathrm{trans}$$

Because of the η- and ξ-rules, it is easy to derive extensionality for both functions and type abstractions:

Proposition A.1

(i) $\Gamma; \Delta \vdash t = s : T_1 \to T_2$ *iff* $\Gamma; \Delta, x : T_1 \vdash t\, x = s\, x : T_2$.

(ii) $\Gamma; \Delta \vdash t = s : \forall X.T$ *iff* $\Gamma, X; \Delta \vdash t\, X = s\, X : T$.

Proof. The left-to-right direction is just the congruence rule. For the other direction, we have

$$t \overset{\eta}{=} \lambda x.t\, x \overset{\xi}{=} \lambda x.s\, x \overset{\eta}{=} s$$

and similarly for type abstractions. □

A.2 Internal language in a λ2-fibration

Every $\lambda 2$-fibration $p : \mathcal{G} \to \mathcal{S}$ has a notion of internal language, which is a polymorphic lambda calculus whose base types $\underline{A}$ in context Γ are given by objects A in $\mathcal{G}_{[\![\Gamma]\!]}$. For every such object A and substitution $\sigma : \Gamma' \mapsto \Gamma$, we identify the types $\Gamma' \vdash \underline{[\![\sigma]\!]^* A}$ type and $\Gamma' \vdash \underline{A}[\sigma]$ type. Base term constants $\underline{f} : \underline{A}$ in context $\Gamma; \Delta$ are given by morphisms $f : [\![\Delta]\!] \to A$ in $\mathcal{G}_{[\![\Gamma]\!]}$. The interpretation is extended to these constants by defining $[\![\underline{A}]\!] = A$ and $[\![\underline{f}]\!] = f$. (Note in particular that $[\![\underline{A}[\sigma]]\!] = [\![\sigma]\!]^* A = [\![\underline{[\![\sigma]\!]^* A}]\!]$, which justifies the identification of $\underline{[\![\sigma]\!]^* A}$ and $\underline{A}[\sigma]$.) We also add constants making internal and external function and forall types agree, i.e., new constants $\Gamma; \Delta \vdash \mathsf{lam}_{\mathsf{A,B}} : (\underline{A} \to \underline{B}) \to \underline{A \Rightarrow B}$ and $\Gamma; \Delta \vdash \mathsf{Lam}_{\mathsf{A}} : (\forall X.\underline{A}) \to \underline{\forall A}$

with equations

$$
\begin{gathered}
\Gamma; \Delta, x : \underline{A \Rightarrow B} \vdash \mathsf{lam}_{A,B}\,(\lambda a\,.\,\underline{\mathsf{ev}}\,(x, a)) = x : \underline{A \Rightarrow B} \\
\Gamma; \Delta, y : A \to B \vdash \lambda a\,.\,\underline{\mathsf{ev}}\,(\mathsf{lam}_{A,B}\,y, a) = y : A \to B \\
\Gamma; \Delta, x : \underline{\forall A} \vdash \mathsf{Lam}_A\,(\Lambda X.\underline{\epsilon_A}\,x) = x : \underline{\forall A} \\
\Gamma; \Delta, y : \forall X.\underline{A} \vdash (\Lambda X.\underline{\epsilon_A}\,(\mathsf{Lam}_A y)) = y : \forall X.\underline{A}
\end{gathered}
$$

Here, $\underline{\mathsf{ev}}$ is the internal term corresponding to the external evaluation map $\mathsf{ev} : (A \Rightarrow B) \times A \to B$ and, similarly, $\underline{\epsilon_A}$ is the term corresponding to the counit $\epsilon : \pi^*\forall \to \mathsf{Id}$ of the adjunction $\pi^* \dashv \forall$. The equations above thus state that $\underline{A \Rightarrow B} \cong (\underline{A} \to \underline{B})$ and $\underline{\forall A} \cong \forall X.\underline{A}$. Semantically, we interpret both $\mathsf{lam}_{A,B}$ and Lam_A as identity morphisms. We will often abuse notation and treat $\underline{A \Rightarrow B}$ and $(\underline{A} \to \underline{B})$ as identical.

Concretely for the internal language of the $\lambda 2$-fibration constructed in Theorem 4.6, the base types in context Γ are given by equality preserving fibred functors $F : |\mathsf{Rel}(U)|^{|\Gamma|} \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ — so that, in particular, closed base types are given by objects in $\mathsf{Rel}(\mathcal{E})$ — and base term constants are given by fibred natural transformations between such functors. This internal language is for objects of $\mathsf{Rel}(\mathcal{E})$, but we can embed an object A of $\mathcal{B}$ as the object $\mathsf{Eq}A$ of $\mathsf{Rel}(\mathcal{E})$: fullness and faithfulness of Eq ensures that morphisms from $\mathsf{Eq}A$ to $\mathsf{Eq}B$ coincide with morphisms from A to B.

A.3 Interpreting the simply typed lambda calculus in a CCC

Because so much of our reasoning uses the cartesian closed structure in each fibre of the $\lambda 2$-fibration, for the sake of completeness we now give the details of the interpretation of the simply typed lambda calculus in a cartesian closed category. In order to settle the notation, recall that a category $\mathcal{C}$ with finite products is *cartesian closed* if the functor $- \times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $A \Rightarrow \_$ for each object A, i.e., for each A, B, there is an object $A \Rightarrow B$ and an isomorphism

$$\phi : \mathsf{Hom}_{\mathcal{C}}(\Gamma \times A, B) \cong \mathsf{Hom}_{\mathcal{C}}(\Gamma, A \Rightarrow B)$$

natural in Γ, i.e., for all $f : \Gamma' \to \Gamma$ and all $h : \mathsf{Hom}_{\mathcal{C}}(\Gamma \times A, B)$, we have $\phi(h) \circ f = \phi(h \circ (f \times \mathsf{id}))$ or, equivalently, $\phi^{-1}(h) \circ (f \times \mathsf{id}) = \phi^{-1}(h \circ f)$. The unit of this adjunction is the "evaluation map" $\mathsf{ev}_{A,B} = \phi^{-1}(\mathsf{id}_{A \Rightarrow B}) : (A \Rightarrow B) \times A \to B$.

The simply typed λ-calculus is interpreted in a cartesian closed category by interpreting each term $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ as a morphism $[\![t]\!] : A_1 \times \ldots \times A_n \to B$ as follows:

$$
\begin{aligned}
[\![x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i]\!] &= \pi_i : A_1 \times \ldots \times A_n \to A_i \\
[\![\Gamma, x : A \vdash t : B]\!] &= [\![\Gamma \vdash t : B]\!] \circ \pi_1 : \Gamma \times A \to B \quad \text{if } t \text{ weakened} \\
[\![\Gamma \vdash \lambda x\,.\,t : A \to B]\!] &= \phi([\![\Gamma, x : A \vdash t : B]\!]) \\
e[\![\Gamma \vdash f\,t : B]\!] &= \mathsf{ev}_{A,B} \circ \langle [\![\Gamma : f : A \to B]\!], [\![\Gamma \vdash t : A]\!] \rangle
\end{aligned}
$$

If the substitution $\sigma : \Delta \mapsto \Gamma$ is interpreted as a morphism $[\![\sigma]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$, then $[\![\Gamma \vdash t[\sigma]]\!] = [\![\Delta \vdash t]\!] \circ \sigma$. It is then easily checked that $[\![(\lambda x\,.\,t)\,a]\!] = [\![t[a/x]]\!]$. In particular, the substitution $[a/x] : (\Gamma, x : A) \mapsto \Gamma$ is given by $\langle \mathsf{id}, [\![a]\!] \rangle : \Gamma \to \Gamma \times A$.

We will often implicitly use the following lemma:

Lemma A.2

(i) $\phi^{-1}(\llbracket \vdash \lambda x : A \,.\, x \rrbracket) = \mathsf{id}_{\llbracket A \rrbracket}$.

(ii) *For all* $\Gamma \vdash f : B \to C$ *and* $\Gamma \vdash g : A \to B$, *we have* $\phi^{-1}(\llbracket \lambda x \,.\, f(g(x)) \rrbracket) = \phi^{-1}(\llbracket f \rrbracket) \circ \langle \pi_1, \phi^{-1}(\llbracket g \rrbracket) \rangle$.

(iii) *In particular, if* Γ *is empty then* $\phi^{-1}(\llbracket \vdash \lambda x \,.\, f(g(x)) \rrbracket) = \phi^{-1}(\llbracket \vdash f \rrbracket) \circ \phi^{-1}(\llbracket \vdash g \rrbracket)$ *up to the isomorphism* $\mathbf{1} \times B \cong B$.

Proof.

(i) $\phi^{-1}(\llbracket \vdash \lambda x : A \,.\, x \rrbracket) = \phi^{-1}(\phi(\llbracket x : A \vdash x \rrbracket)) = \mathsf{id}_{\llbracket A \rrbracket}$.

(ii)
$$\begin{aligned}
\phi^{-1}(\llbracket \Gamma \vdash \lambda x \,.\, f(g(x)) : A \to C \rrbracket) &= \phi^{-1}(\phi(\llbracket \Gamma, x : A \vdash f(g(x)) : C \rrbracket)) \\
&= \mathsf{ev} \circ \langle \llbracket f \rrbracket \circ \pi_1, \mathsf{ev} \circ \langle \llbracket g \rrbracket \circ \pi_1, \pi_2 \rangle \rangle \\
&= \mathsf{ev} \circ \langle \llbracket f \rrbracket \circ \pi_1, \mathsf{ev} \circ (\llbracket g \rrbracket \times \mathsf{id}) \rangle \\
&= \mathsf{ev} \circ \langle \llbracket f \rrbracket \circ \pi_1, \phi^{-1}(\llbracket g \rrbracket) \rangle \\
&= \mathsf{ev} \circ (\llbracket f \rrbracket \times \mathsf{id}) \circ \langle \pi_1, \phi^{-1}(\llbracket g \rrbracket) \rangle \\
&= \phi^{-1}(\llbracket f \rrbracket) \circ \langle \pi_1, \phi^{-1}(\llbracket g \rrbracket) \rangle
\end{aligned}$$

(iii) One part of the isomorphism $\mathbf{1} \times B \cong B$ is given by $\langle !_B, \mathsf{id} \rangle$, where $!_B : B \to \mathbf{1}$ is the unique morphism from B into $\mathbf{1}$. By uniqueness of this morphism, we have $!_{\mathbf{1} \times A} = \pi_1 = !_B \circ \phi^{-1}(\llbracket g \rrbracket)$, so that $\phi^{-1}(\llbracket \vdash \lambda x \,.\, f(g(x)) \rrbracket) = \phi^{-1}(\llbracket \vdash f \rrbracket) \circ \langle !_B, \mathsf{id} \rangle \circ \phi^{-1}(\llbracket \vdash g \rrbracket) \cong \phi^{-1}(\llbracket \vdash f \rrbracket) \circ \phi^{-1}(\llbracket \vdash g \rrbracket)$ simply by instantiating (ii). □

B Proofs

B.1 Proofs from Section 4

Lemma 4.4 *Let* $U : \mathcal{E} \to \mathcal{B}$ *be a bifibration with fibred terminal objects and* $\mathcal{B}$ *be a CCC.*

(i) *If* $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ *has a left adjoint* Q, *then* Eq *preserves exponentials iff* Q *satisfies the Frobenius property. Such a* Q *exists if* $U : \mathcal{E} \to \mathcal{B}$ *has full comprehension,* $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ *is full and* $\mathcal{B}$ *has pushouts.*

(ii) *If* $U : \mathcal{E} \to \mathcal{B}$ *is a fibred CCC and has simple products (i.e., if, for every projection* $\pi_B : A \times B \to A$ *in* $\mathcal{B}$, *the reindexing functor* π_B^* *has a right adjoint and the Beck-Chevalley condition holds), then* $\mathcal{E}$ *is a CCC and* U *preserves the cartesian closed structure.*

Proof.

(i) The first part is Proposition 6.2 in Hermida and Jacobs [14] in the case of *homogenous relations*; the same proof is applicable here. For the second part, if we have comprehension and pushouts, then Q can be defined as mapping a relation R over (A, B) to the pushout of π_1 and π_2, where $\langle \pi_1, \pi_2 \rangle : \{R\} \to A \times B$

is the canonical map:

$$\begin{tikzcd}
\{R\} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d] \\
B \arrow[r] & Q(R)
\end{tikzcd}$$

To see that Q is left adjoint to $\mathsf{Eq} = \Sigma_\delta \circ K$, first note that $\mathsf{Eq} \dashv \{\_\} \circ J$, where $J : \mathsf{Rel}(\mathcal{E}) \to \mathcal{E}$ arises from the pullback construction of $\mathsf{Rel}(\mathcal{E})$:

$$\begin{tikzcd}
\mathsf{Rel}(\mathcal{E}) \arrow[r, shift left, "J"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathsf{Rel}(U)"'] & \mathcal{E} \arrow[l, shift left, "\Sigma_\delta"] \arrow[d, bend right, "U"'] \arrow[d, bend left, "\{\}"] \\
\mathcal{B} \times \mathcal{B} \arrow[r, "_\times_"'] & \mathcal{B} \arrow[u, "\dashv K \dashv"]
\end{tikzcd}$$

Since Eq is always faithful and is full by assumption, the unit of the adjunction $\mathsf{Eq} \dashv \{\_\} \circ J$ is a natural isomorphism, i.e., $\{\mathsf{Eq}(A)\} \cong A$ for all A. By the universal property of the pushout, morphisms $f : Q(R) \to C$ are in bijective correspondence with pairs of morphisms $f_1 : A \to C$ and $f_2 : B \to C$ such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$. This is only the case if the following diagram commutes:

$$\begin{tikzcd}
\{R\} \arrow[r, "f_1 \circ \pi_1"] \arrow[d, "\langle \pi_1, \pi_2 \rangle"'] & \{\mathsf{Eq}(C)\} \arrow[d, "\delta"] \\
A \times B \arrow[r, "f_1 \times f_2"'] & C \times C
\end{tikzcd}$$

But this is exactly the diagram from the definition of full comprehension [17, Def. 10.4.7]. Thus, by full and faithfulness of Eq, this diagram commutes iff there is a morphism $g : R \to \mathsf{Eq}(C)$ such that $\pi_1 \circ f_1 = \{g\}$. In other words, morphisms from $Q(R)$ to C are in bijective correspondence with morphisms from R to $\mathsf{Eq}(C)$, as required.

For the Frobenius property, we need to show that $Q(R \times \mathsf{Eq}\, C) = Q(R) \times C$, i.e., that $Q(R) \times C$ is the pushout

$$\begin{tikzcd}
\{R\} \times C \arrow[r, "\pi_1 \times \mathsf{id}"] \arrow[d, "\pi_2 \times \mathsf{id}"'] & A \times C \arrow[d] \\
B \times C \arrow[r] & Q(R) \times C
\end{tikzcd}$$

Here, we have used the facts that $\{\mathsf{Eq}(C)\} = C$ and that $\{\}$ is a right adjoint and thus preserves products. However, since $\mathcal{B}$ is a CCC, we have that $\_ \times C$ is a left adjoint and thus preserves colimits.

(ii) This is Proposition 9.2.4 in Jacobs [17].

□

Theorem 4.6 *If* $\mathsf{Rel}(U)$ *is an equality preserving arrow fibration and a* $\forall$*-fibration, then there is a* $\lambda 2$*-fibration in which types* $\Gamma \vdash T$ *are interpreted as equality preserving fibred functors* $[\![T]\!] : |\mathsf{Rel}(U)|^{|\Gamma|} \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ *and terms* $\Gamma; \Delta \vdash t : T$ *are interpreted as fibred natural transformations* $[\![t]\!] : [\![\Delta]\!] \to [\![T]\!]$.

Proof. We construct a $\lambda 2$-fibration $p : \mathcal{G} \to \mathcal{S}$ that interprets every System F type judgement $\Gamma \vdash T$ as an equality preserving fibred functor $[\![T]\!] : |\mathsf{Rel}(U)|^{|\Gamma|} \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ and every System F term judgement $\Gamma; \Delta \vdash t : T$ as a fibred natural transformation $[\![t]\!] : [\![\Delta]\!] \to [\![T]\!]$. Let $\mathcal{S}$ be the category with natural numbers as objects and morphisms from n to m consisting of m-tuples of equality preserving fibred functors $H_i : |\mathsf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathsf{Rel}(U)$. Let $\mathcal{G}$ be the category with fibres $\mathcal{G}_n = |\mathsf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ and morphisms between fibres given by $\mathsf{Hom}_{\mathcal{G}}(F, G) = \{(f : n \to m,\ \eta : F \to G \circ f)\}$, where F and G are objects of $\mathcal{G}_n$ and $\mathcal{G}_m$, respectively, and the composition $G \circ f$ is defined componentwise. The functor $p : \mathcal{G} \to \mathcal{S}$ mapping each object F of $\mathcal{G}_n$ to n is a fibration. The base $\mathcal{S}$ has finite products given by natural number addition, and 1 is a generic object Ω for $\mathcal{S}$. Because functors $F : |\mathsf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathsf{Rel}(U)$ in $\mathcal{G}_n$ have discrete domain, their products and exponentials are given pointwise. These pointwise exponentials exist since $\mathsf{Rel}(U)$ is an equality preserving arrow-fibration. Thus, p has fibred finite products and fibred exponentials. Finally, p has simple Ω-products since $\mathsf{Rel}(U)$ is a $\forall$-fibration. Thus, p is a $\lambda 2$-fibration. $\square$

B.2 Proofs from Section 5

Lemma 5.3 *If the underlying bifibration satisfies the Beck-Chevalley condition, then $\langle \_ \rangle : \mathcal{B}^{\to} \to \mathsf{Rel}(\mathcal{E})$ is full and faithful if $\mathsf{Eq} : \mathcal{B} \to \mathsf{Rel}(\mathcal{E})$ is.*

Proof. Assume $f : A \to B$ and $g : A' \to B'$. Given a morphism $(h, k) : f \to g$ in $\mathcal{B}^{\to}$, the morphism $\langle h, k \rangle : \langle f \rangle \to \langle g \rangle$ is defined via the universal property of $\langle f \rangle = (f, \mathsf{id})^* \mathsf{Eq} B$. This ensures that $\langle h, k \rangle$ is over (h, k), and thus that the graph functor is faithful. For fullness, consider $\alpha : \langle f \rangle \to \langle g \rangle$ over (α_1, α_2). The opcartesian definition of $\langle f \rangle$ — given by Lemma 5.2 since the Beck-Chevalley condition is satisfied by assumption — and the cartesian definition of $\langle g \rangle$ give a map $\mathsf{Eq} A \to \mathsf{Eq} B'$. By fullness of Eq, we get a map h such that $g \circ \alpha_1 = \alpha_2 \circ f$ and thus $U\alpha : f \to g$ in $B^{\to}$. The cartesian morphism over (g, id) can then be used to show that α satisfies the universal property defining $\langle U\alpha \rangle$ and thus $\alpha = \langle U\alpha \rangle$ proving fullness. $\square$

Theorem 5.4 (Graph Lemma) *Assume the underlying bifibration satisfies the Beck-Chevalley condition, and let $(F_o, F_r) : \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U)$. For any $h : X \to Y$ in $\mathcal{B}$, there are vertical morphisms $\phi_h : \langle F_o h \rangle \to F_r \langle h \rangle$ and $\psi_h : F_r \langle h \rangle \to \langle F_o h \rangle$ in $\mathsf{Rel}(\mathcal{E})$.*

Proof. The definitions of $\langle F_o h \rangle$ and $\langle h \rangle$ give morphisms $(\mathsf{id}_{F_o X}, F_o h)_{\S} : \mathsf{Eq}\,(F_o X) \to \langle F_o h \rangle$ and $F_r((\mathsf{id}_X, h)_{\S}) : F_r(\mathsf{Eq}\ X) \to F_r \langle h \rangle$. The following diagram commutes:

$$\begin{tikzcd}
(F_o X, F_o X) \arrow[r, "(\mathsf{id}_{F_o X}, F_o h)"] & (F_o X, F_o Y) \\
(F_o X, F_o X) \arrow[u, "(\mathsf{id}_{F_o X}, \mathsf{id}_{F_o X})"] \arrow[r, "(\mathsf{id}_{F_o X}, F_o h)"] & (F_o X, F_o Y) \arrow[u, "(\mathsf{id}_{F_o X}, \mathsf{id}_{F_o Y})"']
\end{tikzcd}$$

Thus, by the universal property of the opcartesian map $(\mathsf{id}_{F_o X}, F_o h)_{\S}$, there is a unique morphism $\phi_h : \langle F_o h\rangle \to F_r\langle h\rangle$ such that the following diagram commutes:

$$\begin{tikzcd}
F_r\,(\mathsf{Eq}\,X) \arrow[r, "F_r((\mathsf{id}_X,h)_\S)"] & F_r\langle h\rangle \\
\mathsf{Eq}\,(F_o X) \arrow[u, "="] \arrow[r, "(\mathsf{id}_{F_o X},F_o h)_\S"'] & \langle F_o h\rangle \arrow[u, dashed, "\exists!\,\phi_h"']
\end{tikzcd}$$

Moreover, ϕ_h is over $(\mathsf{id}_{F_o X}, \mathsf{id}_{F_o Y})$ and thus vertical. A similar argument using the universal property of $(\mathsf{id}_{F_o X}, F_o h)^\S$ gives the existence of a unique vertical morphism $\psi_h : F_r\langle h\rangle \to \langle F_o h\rangle$. □

Lemma 5.5 *Z_o is the carrier of a weak initial F_o-algebra (Z_o, in_o) with mediating morphism $fold_o[A,k]$ and Z_r is the carrier of a weak initial F_r-algebra (Z_r, in_r) with mediating morphism $fold_r[A,k]$.*

Proof. Using the internal language, we first define the term $fold = \Lambda A.\lambda k : \underline{F}A \to A.\lambda z.\,z\,A\,k$. We then define $fold_i[A,k] = \phi^{-1}([\![fold\,\underline{A}\,\underline{k}]\!]_i)$, where ϕ is the bijection corresponding to the adjunction $-\times X \dashv X \Rightarrow -$, and $\underline{A}$ and $\underline{k}$ are the internal expressions corresponding to the components of another F_o- or F_r-algebra (A,k), as appropriate. We further define $(in_o, in_r) = \phi^{-1}([\![\lambda x.\,\Lambda X.\,\lambda k.\,k\,(\underline{t}\,(fold[[\![X]\!],[\![k]\!]])\,x)]\!])$, where $\underline{t}$ is the internal representation of the strength of F .By equational reasoning in System F, $fold_o$ and $fold_r$ are algebra homomorphisms:

$$\begin{aligned}
fold[A,k]\circ in &= \phi^{-1}([\![\lambda z.\,z\,\underline{A}\,\underline{k}]\!])\circ\phi^{-1}([\![\lambda x.\,\Lambda X.\,\lambda k.\,k\,(\underline{t}\,(fold[[\![X]\!],[\![k]\!]])\,x)]\!])\\
&= \phi^{-1}([\![\lambda y.((\lambda z.\,z\,\underline{A}\,\underline{k})(\,\Lambda X.\,\lambda k.\,k\,(\underline{t}\,(fold[[\![X]\!],[\![k]\!]])\,y)))]\!])\\
&= \phi^{-1}([\![\lambda y.(\underline{k}\,(\underline{t}\,(fold[[\![\underline{A}]\!],[\![\underline{k}]\!]])\,y)))]\!])\\
&= \phi^{-1}([\![\underline{k}]\!])\circ\phi^{-1}([\![\underline{t}\,(fold[A,k])]\!])\\
&= k\circ F(fold[A,k])
\end{aligned}$$

□

Lemma 5.6 *Assume that the underlying bifibration satisfies the Beck-Chevalley condition, and that* Eq *is full.*

(i) *If $\mathcal{B}$ is well-pointed, then $fold_o[Z_o, in_o] = \mathsf{id}_Z$.*

(ii) *For every F_o-algebra homomorphism $h : (Z_o, in_o) \to (A, k_A)$, we have that $h \circ fold_o[Z_o, in_o] = fold_o[A, k_A]$.*

Proof.

(i) We want to show $[\![\vdash fold[A,k]]\!]_o = [\![\vdash \lambda z.z]\!]_o$. By the ξ- and η-rules, which are valid in all $\lambda 2$-fibrations [30], it suffices to show that

$$[\![X; k : F_o X \to X \vdash \lambda z\,.\,fold[Z_o, in_o]\,z\,X\,k]\!]_o = [\![X; k : F_o X \to X \vdash \lambda z\,.\,z\,X\,k]\!]_o$$

By well-pointedness, this reduces to showing $[\![\lambda z.\,fold[Z_o, in_o]\,z\,A\,k_A]\!]_o = [\![\lambda z.\,z\,A\,k_A]\!]_o$ for any natural transformation $k : F_o \to \mathsf{Id}$. For this we first prove $fold_o[A, k_A] \circ fold_o[Z_o, in_o] = fold_o[A, k_A]$. The following diagram commutes by weak initiality:

$$\begin{tikzcd} F_o Z_o \arrow[r, "F_o\,(fold_o[A,k_A])"] \arrow[d, "in_o"'] & F_o A \arrow[d, "k_A"] \\ Z_o \arrow[r, "fold_o[A,k_A]"'] & A \end{tikzcd}$$

so $(in_o, k_A) : F_o\,(fold_o[A, k_A]) \to fold_o[A, k_A]$ is a morphism in $\mathcal{B}^{\to}$. The graph functor then gives a morphism $\langle in_o, k_A\rangle : \langle F_o\,(fold_o[A, k_A])\rangle \to \langle fold_o[A, k_A]\rangle$, and thus a F_r-algebra k_1 defined by $\langle in_o, k_A\rangle \circ \psi_{fold_o[A,k_A]} : F_r\langle fold_o[A, k_A]\rangle \to \langle fold_o[A, k_A]\rangle$. By weak initiality of Z_r, the following diagram commutes:

$$\begin{tikzcd} F_r Z_r \arrow[r, "F_r\,(fold_r[\langle fold_o[A,k_A]\rangle,k_1])"] \arrow[d, "in_r"'] & F_r\langle fold_o[A, k_A]\rangle \arrow[d, "k_1"] \\ Z_r \arrow[r, "fold_r[\langle fold_o[A,k_A]\rangle,k_1]"'] & \langle fold_o[A, k_A]\rangle \end{tikzcd}$$

By the IEL, $Z_r = \mathsf{Eq}\,Z_o = \langle \mathsf{id}_{Z_o}\rangle$, so $fold_r[\langle fold_o[A, k_A]\rangle, k_1]$ is actually a morphism from $\langle \mathsf{id}_{Z_o}\rangle$ to $\langle fold_o[A, k_A]\rangle$. Since $fold_r[\langle fold_o[A, k_A]\rangle, k_1]$ is over $(fold_o[Z_o, in_o], fold_o[A, k_A])$, and Eq is always faithful and is full by assumption, Lemma 5.3 gives that $fold_r[\langle fold_o[A, k_A]\rangle, k_1] = \langle fold_o[Z_o, in_o], fold_o[A, k_A]\rangle$ and the following diagram commutes:

$$\begin{tikzcd} Z_o \arrow[r, "fold_o[Z_o,in_o]"] \arrow[d, "\mathsf{id}_{Z_o}"'] & Z_o \arrow[d, "fold_o[A,k_A]"] \\ Z_o \arrow[r, "fold_o[A,k_A]"] & A \end{tikzcd}$$

By the definition of *fold*,

$$\begin{aligned} [\![\lambda z.\, fold[Z_o, in_o]\, z\, A\, k_A]\!]_o &= [\![\lambda z.\, fold[A, k_A]\,(fold[Z_o, in_o]\, z)]\!]_o \\ &= [\![\lambda z.\, fold[A, k_A]\, z]\!]_o \\ &= [\![\lambda z.\, z\, A\, k_A]\!]_o \end{aligned}$$

Thus $fold_o[Z_o, in_o] = \mathsf{id}_{Z_o}$ as required.

(ii) Let $h : (Z_o, in_o) \to (A, k_A)$. The definition of $\langle \_ \rangle$ and the Graph Lemma give a unique morphism $k_1 = \langle in_o, k_A\rangle \circ \psi_k : F_r\langle h\rangle \to \langle h\rangle$, and by weak initiality of Z_r we have $in_r \circ fold_r[\langle h\rangle, k_1] = k_r \circ F_r\,(fold_r[\langle h\rangle, k_1])$. By the IEL, $Z_r = \mathsf{Eq}\,Z_o = \langle \mathsf{id}_{Z_o}\rangle$, so that $fold_r[\langle h\rangle, k_1]$ is a morphism from $\langle \mathsf{id}_{Z_o}\rangle$ to $\langle h\rangle$. Since $fold_r[\langle h\rangle, k_1]$ is over $(fold_o[Z_o, in_o], fold_o[A, k_A])$, Lemma 5.3 gives that $fold_r[\langle h\rangle, k_1] = \langle fold_o[Z_o, in_o], fold_o[A, k_A]\rangle$ and the following diagram also commutes by fullness of the graph functor:

$$\begin{tikzcd} Z_o \arrow[r, "fold_o[Z_o,in_o]"] \arrow[d, "\mathsf{id}_{Z_o}"'] & Z_o \arrow[d, "h"] \\ Z_o \arrow[r, "fold_o[A,k_A]"] & A \end{tikzcd}$$

Thus, $h \circ fold_o[Z_o, in_o] = fold_o[A, k_A,]$.

□

Theorem 5.7 *If the underlying bifibration satisfies the Beck-Chevalley condition, Eq is full, and $\mathcal{B}$ is well-pointed, then (Z_o, in_o) is an initial F_o-algebra.*

Proof. By Lemma 5.5, we know that (Z_o, in_o) is a weak initial F_o-algebra. We must show that $h = fold_o[A, k_A]$ for any $k_A : F_o A \to A$ and any F_o-algebra morphism $h : (Z_o, in_o) \to (A, k_A)$. By Lemma 5.6(ii), $fold_o[A, k_A, =]h \circ fold_o[Z_o, in_o]$ and since $fold_o[Z_o, in_o] = \mathsf{id}_{Z_o}$ by Lemma 5.6(i), we have $h = fold_o[A, k_A]$, as required. □

Lemma B.1 (Surjective pairing) *Assume $\mathcal{B}$ is well-pointed and Eq is full.*

(i) *If $\Gamma; \Delta \vdash t : A \times B$ then $[\![t]\!]_o = [\![t\,(A \times B)\,(\_,\_)]\!]_o$.*

(ii) *If $\Gamma; \Delta \vdash t : A \times B$ then $[\![t]\!]_o = [\![(\pi_1\, t, \pi_2\, t)]\!]_o$.*

Proof.

(i) By functional extensionality $[\![t]\!]_o = [\![t\,(A \times B)\,(\_,\_)]\!]_o$ iff for every X and $f\colon A \to B \to X$, $[\![t\,X\,f]\!]_o = [\![(t\,(A \times B)\,(\_,\_))\,X\,f]\!]_o$. By well-pointedness, it is enough to verify this in the empty context. For every such f, we define $f^* : A \times B \to X$ by $f^* = \lambda t\,.\,t\,X\,f$. For $a\colon A$ and $b\colon B$, we then have $f^*((\_,\_)\,a\,b) = ((\_,\_)\,a\,b)\,X\,f = f\,a\,b$ by the encoding of $(\_,\_)$. Thus, the following diagram commutes:

$$\begin{tikzcd}
[\![A]\!]_o \times [\![B]\!]_o \arrow[r, "(_,_)\,a\,b"] \arrow[d, "[\![f\,a\,b]\!]_o"'] & [\![A \times B]\!]_o \arrow[d, "[\![f^*]\!]_o"] \\
[\![X]\!]_o \arrow[r, "\mathsf{id}"'] & [\![X]\!]_o
\end{tikzcd}$$

This constitutes a morphism in the arrow category. We apply the graph functor to get $\langle [\![(\_,\_)\,a\,b]\!]_o, [\![f\,a\,b]\!]_o \rangle : \mathsf{Eq}([\![\Delta]\!]_o \times [\![A]\!]_o \times [\![B]\!]_o) \to \langle [\![f^*]\!]_o \rangle$. Equivalently, $\langle [\![(\_,\_)]\!]_o, [\![f]\!]_o \rangle : \mathsf{Eq}([\![\Delta]\!]_o) \to ([\![A]\!]_o \Rightarrow [\![B]\!]_o \Rightarrow \langle [\![f^*]\!]_o \rangle)$. Now consider the composition $[\![t]\!]_{\langle [\![f^*]\!]_o \rangle} \circ \langle \mathsf{id}, \langle [\![(\_,\_)]\!]_o, [\![f]\!]_o \rangle \rangle : \mathsf{Eq}([\![\Delta]\!]_o) \to \langle [\![f^*]\!]_o \rangle$. By fullness and faithfulness of the graph functor, the above morphism corresponds the following commuting diagram:

$$\begin{tikzcd}
[\![\Delta]\!]_o \arrow[r, "[\![t\,(A \times B)\,(_,_)]\!]_o"] \arrow[d, "\mathsf{id}"'] & [\![A \times B]\!]_o \arrow[d, "[\![f^*]\!]_o"] \\
[\![\Delta]\!]_o \arrow[r, "[\![t\,X\,f]\!]_o"'] & [\![X]\!]_o
\end{tikzcd}$$

Thus, $[\![t\,X\,f]\!]_o = [\![(t\,(A \times B)\,(\_,\_))\,X\,f]\!]_o$, as desired.

(ii) By extensionality $[\![t]\!]_o = [\![(\pi_1\, t, \pi_2\, t)]\!]_o$ iff $[\![t\,X\,f]\!]_o = [\![(\pi_1\, t, \pi_2\, t)\,X\,f]\!]_o$ for every $f : A \to B \to X$. For every such f we define $g : A \times B \to X$ by $g = \lambda t\,.\,f\,(\pi_1\, t)\,(\pi_2\, t)$. We now use the same arguments as in the proof of (i). Since $g\,((\_,\_)\,a\,b) = f\,(\pi_1\,(a,b))\,(\pi_2\,(a,b)) = f\,a\,b$, we have the morphism $\langle [\![(\_,\_)]\!]_o, [\![f]\!]_o \rangle \colon [\![\Delta]\!]_o \to (\mathsf{Eq}(A) \Rightarrow \mathsf{Eq}(B) \Rightarrow \langle [\![g]\!]_o \rangle)$, as well as a morphism

$\llbracket\Delta\rrbracket_o \to \langle\llbracket g\rrbracket_o\rangle$ corresponding to the following commuting diagram:

$$\begin{tikzcd}
\llbracket\Delta\rrbracket_o \arrow[r, "{\llbracket t\,(A\times B)\,(_,_)\rrbracket_o}"] \arrow[d, "\mathsf{id}"'] & \llbracket A\times B\rrbracket_o \arrow[d, "{\llbracket g\rrbracket_o}"] \\
\llbracket\Delta\rrbracket_o \arrow[r, "{\llbracket t\,X\,f\rrbracket_o}"'] & \llbracket X\rrbracket_o.
\end{tikzcd}$$

By (i), $\llbracket t\rrbracket_o = \llbracket t\,(A\times B)\,(\_,\_)\rrbracket_o$, and the conclusion follows by equational reasoning: $\llbracket(\pi_1\,t, \pi_2\,t)\,X\,f\rrbracket_o = \llbracket f\,(\pi_1\,t)\,(\pi_2\,t)\rrbracket_o = \llbracket g\rrbracket_o \circ \llbracket t\rrbracket_o = \llbracket t\,X\,f\rrbracket_o$.

□

Lemma 5.8 *Assume the underlying bifibration satisfies the Beck-Chevalley condition, and that* Eq *is full.*

(i) *Let* $\Gamma;\Delta \vdash t : \exists X.T$*, let* $\Gamma, Z; \Delta, u : T[Z/X] \vdash s : S$ *and let* $\Gamma;\Delta \vdash f : S \to S'$ *for a closed type* S'*. Then* $\llbracket f(\mathsf{open}\,t\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,s)\rrbracket_o = \llbracket\mathsf{open}\,t\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,f(s)\rrbracket_o$.

(ii) *If* $\Delta;\Gamma \vdash t : \exists X.T$*, then* $\llbracket\mathsf{open}\,t\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,\langle Z,u\rangle\rrbracket_o = \llbracket t\rrbracket_o$.

Proof.

(i) We first define a morphism $\alpha : \llbracket\Delta\rrbracket_r \to \forall(\llbracket T\rrbracket_r \Rightarrow \langle\llbracket f\rrbracket_o\rangle)$ in $\mathsf{Rel}(\mathcal{E})^{|\Gamma|}$ over (h,h'), where $h = \langle\pi_0, \llbracket\Lambda X.\lambda y.s\rrbracket_o\rangle$ and $h' = \llbracket\Lambda X.\lambda y.f(s)\rrbracket_o$. By adjointness of $\forall$ and exponentials, this is equivalent to a morphism $\llbracket\Delta\rrbracket_r \times \llbracket T\rrbracket_r \to \langle\llbracket f\rrbracket_o\rangle$. Since S' is closed, $\llbracket S'\rrbracket_r = \mathsf{Eq}(\llbracket S'\rrbracket_o)$ by the Identity Extension Lemma, and $\llbracket\Lambda X.\lambda y.f(s)\rrbracket_r : \llbracket\Delta\rrbracket_r \times \llbracket T\rrbracket_r \to Eq(\llbracket S'\rrbracket_o)$ is over (h',h'). The following triangle commutes:

$$\begin{tikzcd}
(\llbracket\Delta\rrbracket_o \times \llbracket T\rrbracket_o, \llbracket\Delta\rrbracket_o \times \llbracket T\rrbracket_o) \arrow[d, "{(h,h')}"'] \arrow[dr, "{(h',h')}"] & \\
(\llbracket\Delta\rrbracket_o \times \llbracket S\rrbracket_o, \llbracket S'\rrbracket_o) \arrow[r, "{(\llbracket f\rrbracket_o,\mathsf{id})}"'] & (\llbracket S'\rrbracket_o, \llbracket S'\rrbracket_o)
\end{tikzcd}$$

Thus, the cartesian property of $\langle\llbracket f\rrbracket_o\rangle$ gives a unique $\alpha : \llbracket\Delta\rrbracket_r \times \llbracket T\rrbracket_r \to \langle\llbracket f\rrbracket_o\rangle$ over (h,h'), or equivalently, $\widehat{\alpha} : \llbracket\Delta\rrbracket_r \to \forall(\llbracket T\rrbracket_r \Rightarrow \langle\llbracket f\rrbracket_o\rangle)$. From this, we construct the morphism $\llbracket t\rrbracket_{\langle f\rangle} \circ \langle\mathsf{id}, \widehat{\alpha}\rangle : \llbracket\Delta\rrbracket_r \to \langle\llbracket f\rrbracket_o\rangle$ over $(\llbracket t\,S\,h\rrbracket_o, \llbracket t\,S'\,h'\rrbracket_o)$. Since $\llbracket\Delta\rrbracket_r = \mathsf{Eq}(\llbracket\Delta\rrbracket_o) = \langle\mathsf{id}_{\llbracket\Delta\rrbracket_o}\rangle$, this is in fact a morphism between graph relations, and fullness of the graph functor gives that the following diagram commutes:

$$\begin{tikzcd}
\llbracket\Delta\rrbracket_o \arrow[r, "{\llbracket t\,S\,h\rrbracket_o}"] \arrow[d, "\mathsf{id}"'] & S \arrow[d, "{\llbracket f\rrbracket_o}"] \\
\llbracket\Delta\rrbracket_o \arrow[r, "{\llbracket t\,S'\,h'\rrbracket_o}"'] & S'
\end{tikzcd}$$

By unfolding our encoding of $\mathsf{open}\,t\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,s = t\,S\,(\Lambda Z.\,\lambda u\,.\,s))$, this exactly says that $\llbracket f(\mathsf{open}\,t\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,s)\rrbracket_o = \llbracket\mathsf{open}\,t\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,f(s)\rrbracket_o$.

(ii) By function extensionality, it is enough to show that for every type variable Z and term variable y, we have $\llbracket Z; y : \forall X.T \to Z \vdash (\mathsf{open}\,t\,\mathsf{as}\,\langle X,u\rangle\,\mathsf{in}\,\langle X,u\rangle)\,Z\,y\rrbracket = \llbracket Z; y : \forall X.T \to Z \vdash t\,Z\,y\rrbracket$. In this context, define $f : \exists X.T \to \exists X.T$ by

$f\ p = p\,Z\,y$. We have

$$\begin{aligned}
[\![(\mathsf{open}\ t\ \mathsf{as}\ \langle X, u\rangle\ \mathsf{in}\ \langle X, u\rangle)\ Z\ y]\!]_o &= [\![f(\mathsf{open}\ t\ \mathsf{as}\ \langle X, u\rangle\ \mathsf{in}\ \langle X, u\rangle)]\!]_o \\
&= [\![\mathsf{open}\ t\ \mathsf{as}\ \langle X, u\rangle\ \mathsf{in}\ f(\langle X, u\rangle)]\!]_o \\
&= [\![t\ Z\ y]\!]_o
\end{aligned}$$

Here, the first equality is by the definition of f, the second by (i), and the last one by the definition of open and $\langle \_, \_\rangle$.

□

Lemma 5.9 *W_o is the carrier of a weakly final F_o-coalgebra (W_o, out_o) with mediating morphism $unfold_o[A, k]$ and W_r is the carrier of a weakly final F_r-coalgebra (W_r, out_r) with mediating morphism $unfold_r[A, k]$.*

Proof. The structure of the proof is similar to the proof of Lemma 5.5. We first construct the term $unfold = \Lambda A.\lambda w : A \to \underline{F}A.\lambda x.\langle A, (w, x)\rangle$, and define $unfold_i[A, k] = [\![unfold\ \underline{A}\ \underline{k}]\!]_i$, where $(\underline{A}, \underline{k})$ are the internal expressions corresponding to the F_o- or F_r-coalgebra (A, k), as appropriate. We define the structure maps (out_o, out_r) by $out_i = [\![\lambda z.\mathsf{open}\ z\ \mathsf{as}\ \langle Z, v\rangle\ \mathsf{in}\ (\underline{t}\,(unfold\ Z\ (\pi_1 v)))(\pi_1\, v\,(\pi_2\, v))]\!]_i$, where t is the strength of F. By equational reasoning in System F, $unfold_o$ and $unfold_r$ are coalgebra morphisms:

$$\begin{aligned}
out \circ unfold[A, k] &= [\![\lambda x.\mathsf{open}\ \langle \underline{A}, (\underline{k}, x)\rangle\ \mathsf{as}\ \langle Z, u\rangle\ \mathsf{in}\ \big(\underline{t}(unfold\ Z\ (\pi_1\, u))\ (\pi_1\, u\ (\pi_2\, u))\big)]\!] \\
&= [\![\lambda x.\underline{t}(unfold\ \underline{A}\ \underline{k})\ (\underline{k}\ x)]\!] \\
&= [\![\underline{t}(unfold\ \underline{A}\ \underline{k})]\!] \circ [\![\underline{k}]\!] \\
&= F(unfold[A, k]) \circ k
\end{aligned}$$

□

Lemma 5.10 *Assume the underlying bifibration satisfies the Beck-Chevalley condition, and that* Eq *is full.*

(i) *For every F_o-coalgebra morphism $h : (A, k_A) \to (B, k_B)$ we have $unfold_o[B, k_B] \circ h = unfold_o[A, k_A]$.*

(ii) $unfold_o[W_o, out_o] = \mathsf{id}_{W_o}$.

Proof.

(i) Since h is a coalgebra morphism, $k_B \circ h = F_o h \circ k_A$. Applying the graph functor, we obtain the morphism $\langle k_A, k_B\rangle : \langle h\rangle \to \langle F_o h\rangle$, which by composing with the morphism of the Graph Lemma gives an F_r-coalgebra $\psi_h \circ \langle k_A, k_B\rangle : \langle h\rangle \to F_r\langle h\rangle$. By weak finality, we have a F_r-coalgebra morphism $unfold_r[\langle h\rangle, \psi_h \circ \langle k_A, k_B\rangle] : (\langle h\rangle, \psi_h \circ \langle (k_A, k_B)\rangle) \to (W_r, out_r)$. Since $W_r = Eq(W_o) = \langle \mathsf{id}_{W_o}\rangle$, and since the graph functor is full by Lemma 5.3, we see that $(unfold_o[A, k_A], unfold_o[B, k_B]) : \langle h\rangle \to \langle \mathsf{id}\rangle$ in $\mathcal{B}^{\to}$, i.e., $unfold_o[A, k_A] = unfold_o[B, k_B] \circ h$.

(ii) By function extensionality, it is enough to prove $[\![unfold\ \underline{W_o}\ \underline{out}\ x]\!]_o = [\![x]\!]_o$ for a fresh variable $x : \underline{W_o}$. We first note that by (i), $unfold_o[A, k] = unfold_o[W_o, out] \circ unfold_o[A, k]$ for any A, $k : A \to F(A)$, i.e., by the definition of $unfold_o[A, k]$, for any type X, and terms $h : X \to F(X)$, and $y : X$ we have $[\![\langle X, (h, y),]\rangle]\!] =$

$[\![\mathit{unfold}\,\underline{W_o}\,\underline{\mathit{out}}\,\langle X,(h,y)\rangle]\!]$.

$$\begin{aligned}
[\![x]\!] &= [\![\mathsf{open}\,x\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,\langle Z,u\rangle]\!]\\
&= [\![\mathsf{open}\,x\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,\langle Z,(\pi_1\,u,\pi_2\,u)\rangle]\!]\\
&= [\![\mathsf{open}\,x\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,(\mathit{unfold}\,\underline{W_o}\,\underline{\mathit{out}}\,\langle Z,(\pi_1\,u,\pi_2\,u)\rangle)]\!]\\
&= [\![\mathit{unfold}\,\underline{W_o}\,\underline{\mathit{out}}(\mathsf{open}\,x\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,\langle Z,(\pi_1\,u,\pi_2\,u)\rangle)]\!]\\
&= [\![\mathit{unfold}\,\underline{W_o}\,\underline{\mathit{out}}(\mathsf{open}\,x\,\mathsf{as}\,\langle Z,u\rangle\,\mathsf{in}\,\langle Z,u\rangle)]\!]\\
&= [\![\mathit{unfold}\,\underline{W_o}\,\underline{\mathit{out}}\,x]\!]
\end{aligned}$$

Here, the first equality comes from Lemma 5.8(ii), the second one from Lemma B.1, the third from the observation above, the fourth from Lemma 5.8(i), and the fifth and sixth respectively from Lemma B.1 and Lemma 5.8(ii) again. □

Theorem 5.11 *If the underlying bifibration satisfies the Beck-Chevalley condition, and if* Eq *is full, then* (W_o, out_o) *is a final* F_o*-coalgebra.*

Proof. By Lemma 5.9, (W_o, out_o) is weakly final. We must show that $h = \mathit{unfold}_o[A,k_A]$ for any $k_A : A \to F_oA$ and any F_o-coalgebra morphism $h : (A,k_A) \to (W_o, \mathit{out}_o)$. By Lemma 5.10(i), $\mathit{unfold}_o[A,k_A] = \mathit{unfold}_o[K_o,\mathit{out}_o] \circ h$ and since $\mathit{unfold}_o[K_o,\mathit{out}_o] = \mathsf{id}_{W_o}$, by Lemma 5.10(ii), we have $h = \mathit{unfold}_o[A,k_A]$, as required. □

Theorem 5.13 *Let* $(F_o,F_r),(G_o,G_r) : \mathsf{Rel}(U)^{op} \times \mathsf{Rel}(U) \to_{\mathsf{Eq}} \mathsf{Rel}(U)$*. Further, let* $t^0_A : F_oAA \to G_oAA$ *be a family indexed by objects* A *of* $\mathcal{B}$*, and* $t^1_R : F_rRR \to G_rRR$ *be a family indexed by objects* R *of* $\mathsf{Rel}(\mathcal{E})$ *such that if* R *is over* (A,B)*, then* t^1_R *is over* (t^0A, t^0B)*. Then* t^0 *is a dinatural transformation from* F_o *to* G_o*.*

Proof. Let $g : A \to B$ be a morphism in $\mathcal{B}$. Let $\phi : \mathsf{Eq}\,A \to \langle g\rangle$ and $\psi : \langle g\rangle \to \mathsf{Eq}\,B$ be the maps associated to the opreindexing and reindexing definitions of $\langle g\rangle$. The morphism

$$F_r(\mathsf{Eq}\,B)(\mathsf{Eq}\,A) \xrightarrow{F_r\psi\phi} F_r\langle g\rangle\langle g\rangle \xrightarrow{t^1_{\langle g\rangle}} G_r\langle g\rangle\langle g\rangle \xrightarrow{G_r\,\phi\,\psi} G_r(\mathsf{Eq}\,A)(\mathsf{Eq}\,B)$$

is such that $F_r\,\psi\,\phi$ is over $(F_o(g,\mathsf{id}_A), F_o(\mathsf{id}_B,g))$, $t^1_{\langle g\rangle}$ is over (t^0_A,t^0_B), and $G_r\,\phi\,\psi$ is over $(G_o(\mathsf{id}_A,g), G_o(g,\mathsf{id}_B))$. Since F_r and G_r are equality preserving, $F_r(\mathsf{Eq}\,B)(\mathsf{Eq}\,A) = \mathsf{Eq}\,(F_oBA) = \langle \mathsf{id}_{F_oBA}\rangle$ and $G_r(\mathsf{Eq}\,A)(\mathsf{Eq}\,B) = \mathsf{Eq}\,(G_oAB) = \langle \mathsf{id}_{G_oAB}\rangle$. Finally, by the fullness and faithfulness of the graph functor we have

$$\begin{tikzcd}
F_o\,B\,A \arrow[r, "F_o(g{,}\mathsf{id}_A)"] \arrow[d, "\mathsf{id}"'] & F_o\,A\,A \arrow[r, "t^0_A"] & G_o\,A\,A \arrow[r, "G_o(\mathsf{id}_A{,}g)"] & G_o\,A\,B \arrow[d, "\mathsf{id}"] \\
F_o\,B\,A \arrow[r, "F_o(\mathsf{id}_B{,}g)"'] & F_o\,B\,B \arrow[r, "t^0_B"'] & G_o\,B\,B \arrow[r, "G_o(g{,}\mathsf{id}_B)"'] & G_o\,A\,B
\end{tikzcd}$$

This proves the required hexagon commutes. □