Chapter 8

Short Cut Fusion of Recursive Programs with Computational Effects

Neil Ghani, Patricia Johann¹ Category: Research

Abstract: Fusion is the process of improving the efficiency of modularly constructed programs by transforming them into monolithic equivalents. This paper defines a generalization of the standard build combinator which expresses uniform production of functorial contexts containing data of inductive types. It also proves correct a fusion rule which generalizes the fold/build and fold/buildp rules from the literature, and eliminates intermediate data structures of inductive types without disturbing the contexts in which they are situated. An important special case arises when this context is monadic. When it is, a second rule for fusing combinations of producers and consumers via monad operations, rather than via composition, is also available. We give examples illustrating both rules, and consider their coalgebraic duals as well.

8.1 GENERALIZING SHORT CUT FUSION

8.1.1 Introducing Short Cut Fusion

Fusion is the process of improving the efficiency of modularly constructed programs by transforming them into monolithic equivalents. Short cut fusion [6] is concerned with eliminating list traversals from compositions of components that are "glued" together via intermediate lists. Short cut fusion uses a local transformation — known as the foldr/build rule — to fuse computations which can be written as compositions of applications of the uniform list-consuming function foldr and the uniform list-producing function build given by

¹University of Strathclyde, Glasgow, Scotland

[{]Neil.Ghani,Patricia.Johann}@cis.strath.ac.uk.

```
newtype Mu f = In {unIn :: f (Mu f)}
fold :: Functor f => (f a -> a) -> Mu f -> a
fold h (In k) = h (fmap (fold h) k)
build :: Functor f =>
   (forall a. (f a -> a) -> c -> a) -> c -> Mu f
build g = g In
fold k . build g = g k
```

FIGURE 8.1. The fold and build combinators and fold/build rule.

```
foldr :: (b -> a -> a) -> a -> [b] -> a
foldr c n [] = n
foldr c n (x:xs) = c x (foldr c n xs)
build :: (forall a. (b -> a -> a) -> a -> a) -> [b]
build g = g (:) []
```

The function foldr is standard in the Haskell prelude. Intuitively, foldr c n xs produces a value by replacing all occurrences of (:) in xs by c and the occurrence of [] in xs by n. Thus, sum xs = foldr (+) 0 xs sums the (numeric) elements of the list xs. Uniform production of lists, on the other hand, is accomplished using the combinator build, which takes as input a type-independent template for constructing "abstract" lists and produces a corresponding "concrete" list. Thus, build $(\langle n \rangle - \rangle c \langle 4 \rangle (c \rangle n \rangle)$ produces the list [4,7]. Uniform list transformers can be written in terms of both foldr and build. For example, the function map can be implemented as

map :: (a -> b) -> [a] -> [b]
map f xs = build (\c n -> foldr (c . f) n xs)

The foldr/build rule capitalizes on the uniform production and consumption of lists to improve the performance of list-manipulating programs. It says

foldr c n (build g) = g c n
$$(8.1)$$

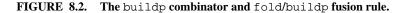
If sqr x = x * x, then this rule can be used, for example, to transform the modular function sum . map $sqr :: [Int] \rightarrow Int$ which produces an intermediate list into an optimized form which does not:

8.1.2 Short Cut Fusion for Inductive Types

Inductive datatypes are fixed points of functors. Functors can be implemented in Haskell as type constructors supporting fmap functions as follows:

VIII–114

```
buildp :: Functor f =>
    (forall a. (f a -> a) -> c -> (a,z)) -> c -> (Mu f, z)
buildp g = g In
fmap (fold k) . buildp g = g k
```



```
class Functor f where
  fmap :: (a -> b) -> f a -> f b
```

The function fmap is expected to satisfy the two semantic functor laws stating that fmap preserves identities and composition. It is well-known that analogues of foldr exist for every inductive datatype. As shown in [4, 5], every inductive type also has an associated generalized build combinator; the extra type c in the type of build is motivated in those papers and to lesser extent in Section 8.3 below. These combinators can be implemented generically in Haskell as in Figure 8.1. There, Mu f represents the least fixed point of the functor f, and In represents the structure map for f, i.e., the "bundled" constructors for the datatype Mu f. The fold/build fusion rule for inductive types can be used to eliminate data structures of type Mu f from computations. The foldr and build combinators for lists can be recovered by taking f to be the functor whose fixed point is [b]. The foldr/build rule can be recovered by taking c to be the unit type as well. As usual, fold and build implement the isomorphisms between inductive types and their Church encodings.

8.1.3 Short Cut Fusion in Context

Short cut fusion handles compositions g . f in which the data structure produced by f is passed from f to q. But what if f produces not just a single data structure, but multiple such structures, embeds these data structures in a non-trivial context, and passes the result to g for consumption of these data structures "in context"? Is it possible to eliminate these intermediate data structures from q. f while keeping the context information, which g may need to compute its result, intact? Standard fusion techniques cannot achieve this: the intermediate data structures produced by f cannot be decoupled from the context in which they are situated. In [2], Fernandes et al. introduce a technique for fusing compositions q. f in which f passes to q not only the intermediate data structure produced by f, but an additional datum as well. Although q requires this datum to compute its result, it is not used when processing the intermediate data structure, and so only the data structure itself needs to be eliminated from q. f. To do this, [2] uses a variant of the standard fold/build rule based on the combinator buildp, which captures the extra datum by returning a data structure embedded in a pair context. The datatype-generic buildp combinator and its associated fold/buildp fusion rule are given in Figure 8.2. There, fmap is the map function

fmap :: (a -> b) -> (a,z) -> (b,z)
fmap f (a,z) = (f a, z)

```
superbuild :: (Functor f, Functor h) =>
  (forall a. (f a -> a) -> c -> h a) -> c -> h (Mu f)
superbuild g = g In
fmap (fold k) . (superbuild g) = g k
```



which witnesses the fact that the type constructor h given by h x = (x, z) is a functor. The context information produced by buildp and used by the consumer in the left-hand side of the fold/buildp fusion rule is reflected in the pair return types of buildp and its template argument, as well as in the mapping of fold across the pair in the associated fold/buildp rule. This rule eliminates intermediate data structures within the context of pairing with an additional datum. But now suppose we want to write a function

gsplitWhen :: (b -> Bool) -> [b] -> [[b]]

which splits a list into sublists at every element that satisfies a given p. Note that the function gsplitWhen splits lists into arbitrary numbers of sublists, depending on the data they contain, and that the type z in the type of buildp cannot be instantiated to allow the return of a number of lists which has the potential to change on each program run. This means that gsplitWhen cannot be written in terms of buildp. Moreover, compositions of gsplitWhen with functions that consume each of the individual "inner" lists produced by gsplitWhen but require the information inherent in its "context list" to compute their results cannot be fused using the fold/buildp rule. But why try to structure programs only with contexts of the form (-,z)? That is, why not consider a generalization of the buildp combinator, and a generalization of the fold/buildp fusion rule which can be used to eliminate intermediate data structures, like those returned by gsplitWhen, which appear in contexts other than just pairs? That is precisely what this paper does. We call these generalizations superbuild and the fold/superbuild rule, respectively. Like buildp and the fold/buildp rule, our superbuild combinator and fold/superbuild fusion rule are available at every inductive datatype. Datatype-generic versions are given in Figure 8.3; note that the type of superbuild is actually generic in both f and h. The generalization of the pair context in the type of buildp is captured by the replacement in the type of superbuild of the type (x, z) by the type h x for a more general "context functor" h. This generalization is further reflected in the replacement of the fmap function for pairs in the fold/buildp rule by the fmap function for the more general context functor h in the fold/superbuild rule. The fold combinator in the fold/superbuild rule is the one for Mu f, as usual. These fmap and fold functions are guaranteed to be defined precisely because the type of superbuild requires both f and h to be functors. We argue in Section 8.3 that the fold/superbuild rule holds for a large class of functors h.

Taking h x = x gives the generalized build combinator and fold/build rule from Figure 8.1, while taking h x = (x, z) gives the buildp combina-

tor and fold/buildp rule from Figure 8.2. In general, the fold/superbuild rule can fuse compositions in which context information describable by non-pair functors is passed, along with intermediate data structures, from producer to consumer. Indeed, the fold/superbuild rule eliminates intermediate structures of type Mu f obtained by mapping a consumer expressed as a fold over the data of type Mu f stored in a context specified by a functor h. Thus, setting c = [b], h = x = [x], and f to be the functor whose least fixed point is [b], we can write

If lgh = foldr $(x \rightarrow (1+))$ 0 then using the fold/superbuild rule to fuse the composition evLghs = map lgh . gsplitWhen even gives

Note that evLghs' trades production and consumption of the list of intermediate lists returned by gsplitWhen even in evLghs for production of the corresponding list of values obtained by applying lgh to each such list.

8.1.4 Short Cut Fusion in Effectful Contexts

The ability to fuse intermediate data structures in context turns out to be the key to extending short cut fusion to the effectful setting. Although fusion in the presence of computational effects has been studied by other researchers (see, e.g., [10, 12, 14, 16]), short cut fusion in particular has not previously been formally explored in this context. To perform short cut fusion in an effectful context, the functional argument to superbuild, and thus superbuild itself, must have a monadic return type. Monads can be implemented in Haskell as type constructors supporting >>= and return operations as follows; these operations are expected to satisfy the semantic monad laws.

```
class Monad m where
  return :: a -> m a
  (>>=) :: m a -> (a -> m b) -> mb
```

If m is a monad, setting h to m in superbuild's type gives the msuperbuild combinator in Figure 8.4. The accompanying fold/msuper build rule is the natural "monadification" of the fold/superbuild rule; we give an example of its use in Section 8.2. As we see in Section 8.3.3 the fold/msuper build rule

```
msuperbuild :: (Functor f, Monad m) =>
  (forall a. (f a -> a) -> c -> m a) -> c -> m (Mu f)
msuperbuild g = g In
msuperbuild g c >>= fold k = g k c >>= id
```

FIGURE 8.4. The msuperbuild combinator and fold/msuperbuild fusion rule.

follows from the fold/super build rule and standard properties of monad multiplication. This rule does not eliminate the monadic context described by m, but does eliminate intermediate data structures of type Mu f within that monadic context. Moreover, although the rule does not change the type of the context containing the data structure, it can change the context itself, and so is more sophisticated than its non-monadic counterpart.

The remainder of this paper is structured as follows. In Section 8.2 we apply our new fold/superbuild and fold/msuperbuild rules to substantive examples. In Section 8.3 we show how the superbuild and msuperbuild combinators are derived from initial algebra semantics, and prove the correctness of their associated fusion rules. In Section 8.4 we give non-monadic and monadic superdestroy/unfold rules dual to our non-monadic and monadic fold/super build rules; our results for superbuild and msuperbuild are easily dualized to prove them correct. In Section 8.5 we discuss related work, and in Section 8.6 we conclude and offer directions for future research. A Haskell implementation of our results and an additional example highlighting the versatility of our rules are available at http://www.cs.nott.ac.uk/~nxg.

8.2 EXAMPLES

In this section we give some more sophisticated examples showcasing the power of the fold/superbuild and fold/msuperbuild fusion rules. Our first example shows that the fold/superbuild rule can be used to eliminate intermediate data structures other than lists. Our second example shows that the fold/msuperbuild rule can eliminate data structures within the state monad.

Example 8.1. Consider the simple arithmetic expression datatype given by

data Opr = Add | Mul | Sub deriving (Eq, Show)

data Expr = Lit Int | Op Opr Expr Expr deriving (Eq, Show)

The fold combinator for expressions, the instance of superbuild for expressions where c is Expr and h x is [x], and the associated fusion rule are

foldExpr :: (Int -> a) -> (Opr -> a -> a -> a) -> Expr -> a
foldExpr l o e = case e of
 Lit i -> l i
 Op op el e2 -> o op (foldExpr l o el)

```
VIII-119
```

```
(foldExpr l o e2)
superbuildExpr :: (forall a. (Int -> a) ->
    (Opr -> a -> a -> a) -> Expr -> [a]) -> Expr -> [Expr]
superbuildExpr g = g Lit Op
map (foldExpr l o e) (superbuildExpr g) = g l o e
  If we define opToHas Add = (+), opToHas Mul = (*), and opToHas
Sub = (-), then we can implement an interpreter which traces the evaluation
steps taken in computing the integer values represented by expressions as
trace :: Expr -> [Expr]
trace = superbuildExpr g
g :: (Int -> a) -> (Opr -> a -> a -> a) -> Expr -> [a]
gloe = case e of
           Lit i -> [l i]
           Op op e1 e2 -> let b1 = foldExpr l o e1
                                b2 = foldExpr l o e2
                                e' = o op bl b2
                               Lit k = last (g Lit Op el)
                                Lit j = last (g Lit Op e2)
                                bls = g l o el
                                b2s = g l o e2
                             in (e' : (map (x \rightarrow 0 op x b2)
                                    (tail bls))
                                 ++ (map (o op (last bls))
                                    (tail b2s))
                                 ++ [l (opToHas op k j)])
For example, if
myexp = Op Mul (Op Add (Lit 5) (Lit 6))
                (Op Sub (Lit 7) (Lit 4))
then trace myexp generates the trace
[ Op Mul (Op Add (Lit 5) (Lit 6)) (Op Sub (Lit 7) (Lit 4)),
```

Op Mul (Lit 11) (Op Sub (Lit 7) (Lit 4)), Op Mul (Lit 11) (Lit 3), Lit 33]

Once an interpreter trace is built, we can perform various analyses of it. For example, we can measure the computational effort required to compute the value represented by each expression arising in the evaluation of a given expression. For this we use count, which counts 0 units of effort to compute a literal, 2 to perform an addition, 3 to perform a subtraction, and 5 to perform a multiplication.

We then have

```
costExprs :: Expr -> [Int]
costExprs expr = map (foldExpr (\x -> 0) count)
```

```
(superbuildExpr g expr)
```

Thus costExprs myexp generates the result [10,8,5,0]. Fusion using the fold/superbuild rule gives costExprs' g ($x \rightarrow 0$) count — an equivalent function in which no intermediate list of expressions is constructed.

Example 8.2. Pardo [14] shows that graph traversal algorithms, such as depth-first and breadth-first traversal, can be written as calls to a monadic unfold combinator. We show that these algorithms can be written in terms of msuper build. The relationship between monadic and non-monadic unfold combinators, and between superbuild and msuperbuild, is discussed in Section 8.5 below.

A graph traversal is represented as a function which takes as input a list of root vertices of a graph and returns a list containing the vertices met in order as the graph is traversed. We can represent the vertices of a graph by integers, and a graph by an adjacency list function for vertices as follows:

```
type V = Int
type Graph = V -> [V]
```

In a graph traversal, each vertex is visited at most once. To avoid repeated visits to vertices we can use the state monad [13, 15] to maintain a list of vertices visited previously in the computation and thread this list through the traversal. We therefore define a datatype of *visit-dependent data*, each element of which is a function taking a list of vertices already visited as input and returning a datum depending on that list together with an updated list of visited vertices. We have

```
data State s a = State {runstate :: s -> (s,a)}
```

```
type Vis a = State [V] a
```

Visit-dependent data support the following useful auxiliary functions:

```
data Unit = Unit
emp :: Vis a -> a
emp xs = snd (runstate xs [])
sunion :: V -> Vis Unit
sunion v = State (\vs -> (v:vs, Unit))
mem :: V -> Vis Bool
mem v = State (\vs -> (vs, elem v vs))
```

With this machinery we can define depth-first traversal as in Figure 8.5. There, dft first allocates an empty list of visited vertices, then runs depthFirst, yield-ing a final list of visited vertices, and then de-allocates this visitation list and

VIII–120

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VIII-121
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```
dft :: (V -> [V]) -> [V] -> [V]
dft g vs = emp (depthFirst g vs)
depthFirst :: (V -> [V]) -> [V] -> Vis [V]
depthFirst g = msuperbuild (df g)
df :: (V -> [V]) ->
forall a. (V -> a -> a) -> a -> [V] -> Vis a
df g c n vs = case vs of
[] -> return n
(x:xs) -> mem x >>=
(\b -> if b then df g c n xs
else sunion x >>=
(\z -> df g c n (g x ++ xs) >>=
(\ys -> return (c x ys))))
```

FIGURE 8.5. Depth-first graph traversal functions.

returns the list resulting from the traversal. At each iteration of the traversal, df explores the current list of roots in v_S to find a vertex it has not reached before. This is done by removing from the front of v_S all vertices for which mem x is true until either an unvisited vertex or the end of v_S is encountered. When an unvisited vertex x is encountered, df adds x to the list of vertices visited, recursively computes the depth-first traversals of the graphs rooted at x's children, as well as those specified by the rest of the vertices in v_S , and then returns the list of vertices obtained by adding x to the list of vertices recording the order in which the rest of the vertices are traversed. The code for breath-first search is identical, except that the function bf corresponding to df uses $x_S ++ g x$ rather than $g x ++ x_S$. To traverse a particular graph we specify the desired traversal, the graph's adjacency list function, and its root vertices. For example, if the graph *G* is modeled by g = [2,1], g = [1], and g = [x+1], then depthFirst g [0] computes the depth-first search of *G* starting at root vertex 0.

For example, to consume the result of a traversal with filtergph odd where

we can write one of the following, depending on the desired traversal

dfFil g = emp (depthFirst g [0] >>= filtergph odd)

bfFil g = emp (breadthFirst g [0] >>= filtergph odd)

To perform the same computations without constructing the intermediate lists of visit-dependent vertices, we can use the fold/msuperbuild rule to get

dfFil' g = emp ((df g) fn (return []) [0] >>= id)

bfFil' g = emp ((bf g) fn (return []) [0] >>= id)

Note that the lists obtained by taking any non-empty initial segments of the results of dfFil g and bfFil g — and thus of dfFil' g and bfFil' g — reflect the distinction between the underlying depth-first and breadth-first traversals.

8.3 CORRECTNESS

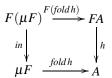
8.3.1 Categorical Preliminaries

Let \mathscr{C} be a category and F be an endofunctor on \mathscr{C} . An *F*-algebra is a morphism $h: FA \to A$ in \mathscr{C} . The object A is called the *carrier* of the *F*-algebra. The *F*-algebras for a functor F are the objects of a category called the *category of F*-algebras and denoted $F \cdot \mathscr{A}lg$. In the category of *F*-algebras, a morphism from $h: FA \to A$ to $g: FB \to B$ is a morphism $f: A \to B$ such that the following diagram commutes:



We call such a morphism an *F*-algebra morphism. If the category of *F*-algebras has an initial object then Lambek's Lemma ensures that this *initial F*-algebra is an isomorphism, and thus that its carrier is a fixed point of *F*. Initiality ensures that the carrier of the initial *F*-algebra is actually a *least* fixed point of *F*. If it exists, the least fixed point for *F* is unique up to isomorphism. Henceforth we write μF for the least fixed point for *F* and *in* : $F(\mu F) \rightarrow \mu F$ for the initial *F*-algebra.

Within the paradigm of initial algebra semantics, every datatype is the carrier μF of the initial algebra of a suitable endofunctor F on a suitable category \mathscr{C} . The unique F-algebra morphism from *in* to any other F-algebra $h: FA \to A$ is given by the interpretation *fold* of the fold combinator for the interpretation μF of the datatype Mu F. The *fold* operator for μF thus makes the following commute:



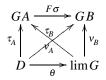
From this diagram, we see that *fold* has type $(FA \rightarrow A) \rightarrow \mu F \rightarrow A$ and that *fold* h satisfies *fold* h (*in* t) = h (F (*fold* h) t). The uniqueness of the mediating map ensures that, for every *F*-algebra h, the map *fold* h is defined uniquely.

As shown in [5], the carrier of the initial algebra of an endofunctor F on \mathscr{C} can be seen not only as the carrier of the initial F-algebra, but also as the limit of the forgetful functor $U_F: F - \mathscr{A}lg \to \mathscr{C}$ mapping each F-algebra $h: FA \to A$ to A.

If $G: \mathcal{C} \to \mathcal{D}$ is a functor, then a *cone* $\tau: D \to G$ to the base *G* with vertex *D* is an object *D* of \mathcal{D} and a family of morphisms $\tau_C: D \to GC$, one for every object *C* of \mathcal{C} , such that for every arrow $\sigma: A \to B$ in $\mathcal{C}, \tau_B = G\sigma \circ \tau_A$ holds.



We usually refer to a cone simply by its family of morphisms, rather than the pair comprising the vertex together with the family of morphisms. A *limit* for $G : \mathscr{C} \to \mathscr{D}$ is an object $\lim G$ of \mathscr{D} and a limiting cone $v : \lim G \to G$, i.e., a cone $v : \lim G \to G$ with the property that if $\tau : D \to G$ is any cone, then there is a unique morphism $\theta : D \to \lim G$ such that $\tau_C = v_C \circ \theta$ for all $C \in \mathscr{C}$.



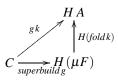
The characterization of μF as $\lim U_F$ provides a principled derivation of the interpretation *build* of the build combinator for μF which complements the derivation of its *fold* operator from standard initial algebra semantics. It also guarantees the correctness of the standard fold/build rules. Indeed, the universal property that the carrier μF of the initial *F*-algebra enjoys as $\lim U_F$ ensures:

- The projection from the limit μF to the carrier of each *F*-algebra defines the *fold* operator with type $(FA \rightarrow A) \rightarrow \mu F \rightarrow A$.
- Given a cone θ : C → U_F, the mediating morphism from it to the limiting cone v : lim U_F → U_F defines a map from C to lim U_F. Since a cone to U_F with vertex C has type ∀x.(Fx → x) → C → x, this mediating morphism defines the *build* operator with type (∀x. (Fx → x) → C → x) → C → μF.
- The correctness of the fold/build fusion rule then follows from the fact that *fold* after *build* is a projection after a mediating morphism, and thus is equal to the cone applied to the specific algebra. Diagrammatically, we have



8.3.2 Correctness of the fold/superbuild Rule

To prove correctness of our fold/superbuild rule we are actually interested in the following variation of the preceding diagram:

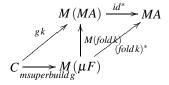


Here, *superbuild* is the interpretation in \mathscr{C} of superbuild. If the functor $H : \mathscr{D} \to \mathscr{E}$ preserves limits — i.e., if, for every functor $G : \mathscr{C} \to \mathscr{D}$ and every limiting cone $v : \lim G \to G$, the cone $Hv : H(\lim G) \to H \circ G$ is also a limit, henceforth denoted $\lim (H \circ G)$ — then this is the diagram for the universal property of $\lim (H \circ U_F)$. We thus ask which functors H preserve limits. It is well-known that right adjoints preserve limits, but this is a more restrictive class of functors than we would like. On the other hand, H needn't preserve *all* limits, just $\lim U_F$.

A connected category is a non-empty category whose underlying graph is connected. A connected limit is a limit of a functor whose domain is a connected category. The limit $\lim U_F : F \cdot \mathscr{A}lg \to \mathscr{C}$ is a connected limit since the category of F-algebras is connected (there is a morphism from the initial F-algebra $in: F(\mu F) \rightarrow \mu F$ to any other *F*-algebra), so knowing that the functor *H* interpreting the type constructor h in the type of superbuild preserves connected limits is sufficient to ensure correctness of the fold/superbuild rule. It is wellknown that strictly positive functors preserve connected limits [3, 7]; in particular, all polynomial functors preserve them. More generally, all functors created by containers preserve connected limits [7]. The class of containers includes functors, such as those whose least fixed points are nested types, which are not strictly positive; the above proof thus covers many situations that are interesting in practice. To prove correctness of the fold/superbuild rule for functors H which do not preserve connected limits, it should be possible to give a formal argument based on logical relations [1]. However, a proof based upon logical relations would not cover examples such such as nested types which preserve connected limits but are not definable in the underlying type theory of the logical relation.

8.3.3 Correctness of the fold/msuperbuild Rule

To see that the fold/msuperbuild rule is correct, we consider the diagram



where M is the interpretation of m in the type of msuperbuild, *bind* and *return* are the interpretations of the >>= and return operations for m, respectively, and

 $f^*x = bindx f$. Correctness of the fold/msuperbuild rule is exactly commutativity of the diagram's outer parallelogram. The diagram's left-hand triangle commutes because it is an instance of the previous diagram, and standard properties of monads ensure that its right-hand side commutes as well. Then

 $g \ k \ c \gg = id = id^* (g \ k \ c)$ $= (id^* \circ g \ k) \ c$ $= ((fold \ k)^* \circ msuperbuild \ g \ c)$ $= (fold \ k)^* (msuperbuild \ g \ c)$ $= msuperbuild \ g \ c \gg = fold \ k$

It is worth noting here that many monads that arise in applications — including the exceptions monad, the state monad, and the list monad — preserve connected limits. The continuations monad, however, does not.

8.4 DUALITY

Our fold/superbuild and fold/msuperbuild rules dualize to the coinductive setting. Shortage of space prevents us from giving the corresponding constructs and results in detail here, so we simply present their implementation. We have

```
unfold :: Functor f => (a -> f a) -> a -> Mu f
unfold k x = In (fmap (unfold k) (k x))
superdestroy :: (Functor f, Functor h) =>
  (forall a. (a -> f a) -> h a -> c) -> h (Mu f) -> c
superdestroy g = g unIn
superdestroy g . fmap (unfold k) = g k
```

When c is Mu f, superdestroy returns an h-algebra which stores coalgebraic f-data. When h is a comonad, i.e., an instance of the Comonad class

class Comonad cm where coreturn :: cm a -> a (=<<) :: cm b -> (cm b -> a) -> cm a

we have

cmsuperdestroy :: (Functor f, Comonad cm) =>
 (forall a. (a -> f a) -> cm a -> c) -> cm (Mu f) -> c
cmsuperdestroy g = g unIn

cmsuperdestroy g (x =<< unfold k) = g k (x =<< id)

8.5 RELATED WORK

The work most closely related to ours is that of Pardo and his coauthors. Like this paper, [14] also investigates conditions under which the composition of a function producing an expression of type $M(\mu F)$ for M a monad and F a functor, and a

function fold k of type $\mu F \rightarrow A$ can be fused to produce an expression of type MA. But there are several crucial differences with our work. First, Pardo uses unfold rather than msuperbuild to construct the intermediate expression. This gives his fusion rule some additional logical generality over ours, since unfold can construct elements of its associated functor f's final coalgebra which are not in f's initial algebra, whereas msuperbuild can construct only elements of f's initial algebra. But when the initial and final algebras of each functor coincide, as in Haskell, this added logical generality yields no advantage in practice.

Secondly, Pardo's monadic hylofusion (and hylofusion in general) is only known to be correct in algebraically compact categories, i.e., categories in which the initial algebra and final coalgebra for each functor coincide. By contrast, our fold/superbuild rule is correct in any category supporting a parametric interpretation of forall, and this condition is independent of any compactness condition. The requirement that the interpreting category be algebraically compact is unfortunate since it generates strictness conditions that must be satisfied, and also requires the underlying monad to be strictness-preserving. This results in strictness condition propagation. By contrast, neither our fold/superbuild nor our fold/msuperbuild rules require the satisfaction of side conditions.

Thirdly, Pardo trades a composition of an unfold and a monadic fold for the computation of an equivalent fixed point. By contrast, our fold/msuperbuild rule trades a bind of a call to msuperbuild with a monadic fold for the bind of the application of the function argument to msuperbuild to the fold's algebra with the identity function. Like all generalizations of the fold/build rule, our fold/msuperbuild rule requires "payment up front" in that the producer in a composition to be fused must be expressed in terms of msuperbuild. (This is not very different from the price paid by expressing consumers in terms of unfold). But our rule delivers a fused result which is simpler than that obtained using Pardo's technique. In particular, the functions obtained from our fusion rules involve only binds of applications involving data structure "templates", rather than fixed point calculations. Their computation is thus guaranteed to terminate.

Finally, Pardo requires the existence of a distributivity law of the underlying monad over the underlying functor to construct the lifting of functors to the Kleisli category on which his monadic hylofusion rule depends. But distributivity laws for arbitrary functors, even those admitting fixed points, need not exist.

Recently, Manzino and Pardo [11] have proposed a fusion rule similar to the fold/msuperbuild rule given here. This rule seems to be interderivable with ours in the presence of distributivity. Meijer and Jeuring [12] have also developed fusion laws in the monadic setting, including a short cut fusion law for eliminating intermediate structures of type FA in a monadic context M. Many fusion methods, including those of [12] and [14], eliminate data structures in the carriers of initial algebras for only restricted classes of functors. By contrast, our method can eliminate data structures of Any inductive type, and can handle non-monadic contexts as well. In addition, Jürgensen [10] and Voigtländer [16] have each defined fusion combinators based on the uniqueness of the map from a free monad to any other monad. These techniques give very different forms of fusion from ours.

8.6 CONCLUSION AND DIRECTIONS FOR FUTURE WORK

In this paper we defined a superbuild combinator which generalizes the standard build combinator and expresses uniform production of functorial contexts containing data of inductive types. We also proved correct a fold/super build fusion rule which generalizes the fold/build and fold/buildp rules from the literature, and eliminates intermediate data structures of inductive types without disturbing the contexts in which they are situated. An important special case arises when this context is monadic. When it is, our fold/msuperbuild rule fuses combinations of producers and consumers via monad operations, rather than via composition. We have given examples illustrating both the fold/superbuild and fold/msuperbuild rules, and considered their coalgebraic duals as well.

The standard fold combinator can consume data structures in any context describable by a functor, but the algebra it uses cannot depend on the context in a non-trivial way. By contrast, context information can be used by algebras to partially determine how the pfold combinator given in [2] will consume the data structures, but unfortunately the contexts are limited to pairs. Interestingly, the pfold/buildp rule given there for context-dependent folds derives from the fold/buildp rule from Figure 8.2 for standard folds. As already noted, it is the fold/buildp rule that our fold/superbuild and fold/msuperbuild rules generalize. One direction for future work is to generalize these rules even further to accommodate *both* context-dependent algebras and non-pair contexts.

Another direction for future work is suggested by considering an even more monadic fusion rule based on fold- and build-like combinators which manipulate algebra-like functions of type f a \rightarrow m a. Such a rule would produce intermediate data structures using "templates" based on so-called monadic algebras and, in the presence of a distributivity rule delta for m over f, would consume data structures using them via a monadic mafold combinator. We'd have

Although a datatype-generic masuperbuild combinator is not defined in [12], several instances of the above fusion rule are given. Yet no correctness proofs for any of these specific instances — let alone any formulation of, or correctness proof for, a datatype-generic fusion rule — are given. We believe an independent proof of the mafold/masuperbuild rule similar to those in Section 8.3 is possible. Although it is not entirely clear how such a proof would go, a proof for monads which preserve connected limits will likely require independent verification that $\lim(MU_{FM}) = M(\mu F)$ for the forgetful functor U_{FM} mapping each

monadic algebra $h: F a \rightarrow M a$ to a, and a proof for monads which do not preserve connected limits will likely be based on logical relations.

The facts that mafold is defined in terms of fold and that masuperbuild g can be expressed as msuperbuild $(\k -> g (return . k))$ together suggest that the mafold/masuperbuild rule might be derivable from (distributivity and) the fold/msuperbuild rule. However, we believe the two rules to offer distinct fusion options in the presence of distributivity; it would be interesting to see which is more useful for programs that arise in practice.

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