2D universes hand in

Dr. Sarah's MAT 1010: Introduction to Mathematics

Geometry of the Earth and Universe

How we measure and view the world around us and decide what is the nature of reality.

submission instructions: Must be completed on this handout and collated into one single PDF for submission in the 2D universes hand in assignment

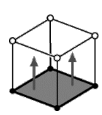
goals:

- Develop problem solving and analysis skills in recognizing patterns and similarities in geometric representations to work towards becoming logical, flexible, critical thinkers and problem solvers.
- Compare and contrast small-scale and large-scale mathematical regions.
- Communicate geometric information in written documents.

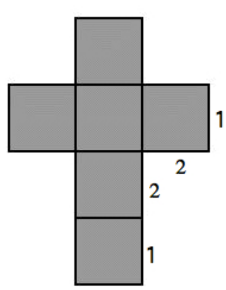
Living in a 2D World

1. How could a 2D Marge Simpson and 2D Lisa Simpson still "pass" each other if they live on an infinite 2D plane, even though they can't walk behind each other (since their surface has no depth and they would bump into each other)?

2. In order to explain a cube to 2D folks and to Homer Simpson, who is trapped in the "third" dimension, a (supposedly) 2D Professor Frink says:
   Frink: – but suppose we exte-end the square beyond the two dimensions of our universe (along the hypothetical z-axis, there).
   Everyone: [gasps]
   Frink: This forms a three-dimensional object known as a "cube", or a "Frinkahedron" in honor of its discoverer, n'hey, n'hey. [Taken from text transcript of 3D Homer segment and Did You Notice? by James A. Cherry]

   Assume the shaded portion on  is the square that is referenced above. Label a $z-$axis on this "Frinkahedron." Image: Davide P. Cervone http://www.math.union.edu/ dpvc/math/4d/welcome.html

3. Professor Frink details one way to form a cube and explain it to others. There is an alternative way of forming a cube—by gluing edges of the figure below together. Label gluing instructions to show which sides you would glue together in order to form a cube. I have started you off by labeling a set of 1s to be glued to each other, and a set of 2s. You will need to glue 5 more sets of edges together, so give the instructions by labeling two 3s that glue together, 4s, 5s, 6s, and 7s. We could give these gluing instructions to Marge or Arthur Square and explain that while the figure can't be glued in 2D, there is enough room to perform the gluing in 3D.

   Label 3s, 4s, 5s, 6s, and 7s on 

   If you are stuck, you can cut out a similar figure, and experiment with folding!

Escher's Representation of Hyperbolic Geometry/Saddle Geometry

In the geom intro hand in, you saw that parallels behave differently in the work below because we can find more parallels to a given "line" through a point off the "line." We'll continue exploring the geometry:

Circle Limit 4: Heaven and Hell by M.C. Escher, 1960

4. Pick an angel and label the tip of the angel's feet, and the two tips of her wings.

5. We can figure out the angle at each of these three points by taking 360° and dividing by the total number of angels and demons that meet at that point, since each creature is supposed to take an equal amount of angle, fitting into 360°. For example, first check that at the feet there are 6 angels and demons coming into that point. So the angle at that point is $\frac{360^\circ}{6}$ =?

6. A different number of creatures come in at each one the wingtips—how many creatures at a wingtip?

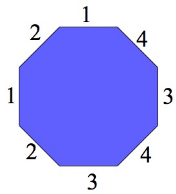
7. What is the angle at each of her wingtips? Compute $\frac{360}{\text{number of creatures coming in}}^\circ =$

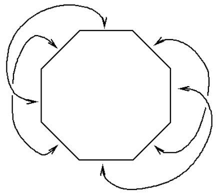
8. What is the sum of the three angles (one at the feet and two at the wingtips)?

9. Lines should preserve symmetry, so they should cut creatures in half (like mirror reflections) and go through the middle of the head, body, and feet. Use this idea to draw some "lines" in this space. Start by drawing mirror "lines" through the center of an angel that cuts her in half and continue these mirrors in both directions through other creatures. Draw at least five "lines" in the work, with at least 2 going through curvy angels and curvy demons.

Holy Donut–Another Hyperbolic Universe

10. One 2D universe from the *The Heart of Mathematics* readings is a 2-holed donut. To form it in a way a 2D creature could somewhat understand, on an octagon we glue the side with a number on it with the side that has the same number on it. It is an exercise in visualization skills to see that the resulting figure is a 2-holed donut:



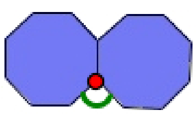



Images: Sascha Rogmann http://www.rogmann.org/math/tori/torus2en.html

Search the web to find the measure of one interior angle of a flat octagon. What is it?

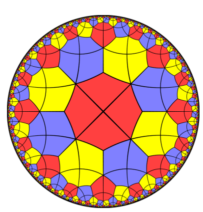
11. To understand why the laws of Euclidean geometry that you learned in high school do not apply to the 2-holed donut, we can look to see whether octagons will tile the plane or not. So we would like to know whether we can take a certain number of octagons (instead of angels and demons like Escher used in the distorted hyperbolic *Heaven and Hell* work...) and put them together around a vertex in order to form 360 degrees. First, if we put two of them together, and want to understand how much angle they take up, we can double a single interior angle of a flat octagon—since they each take up the same amount of space. So double your angle from your response in #10:

12. How much of 360 degrees is left over at the red point when two octagons are placed side by side—the leftover angle is indicated by the green arc ($360° -$ response from #11)?



13. What happens if we try and place three octagons together at a vertex? Could they fit into 360 degrees?

14. We can create a 2-holed donut by using distorted octagons with 45 degree interior angles that fit together to tile hyperbolic space. Eight of these glue together like in Escher's work to form 360 degrees at a vertex and so they tile the space ($45 \times 8 = 360$). Now we understand some of the issues that Escher faced, why his *Heaven and Hell* work looked like it did, and why these spaces are not flat—in hyperbolic geometry sides bow in to be able to fit more together.

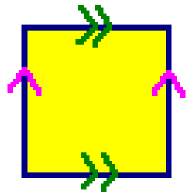
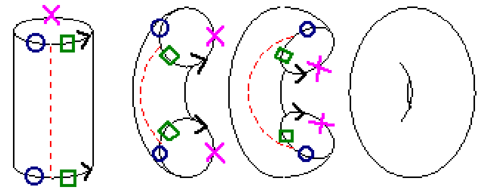



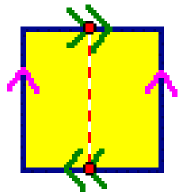
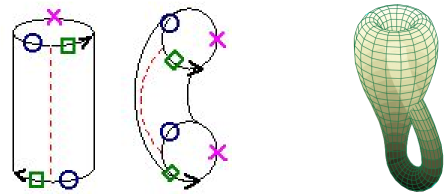
CC-BY 2.5 by Claudio Rocchini

Circle Limit 4: Heaven and Hell by M.C. Escher, 1960

Klein Bottle Tic-Tac-Toe–A Euclidean Universe

15. As in the 2D universes intro video, the torus and Klein bottle can be created by gluing edges of a square together, straight across for the torus, and with one twist for the Klein bottle:

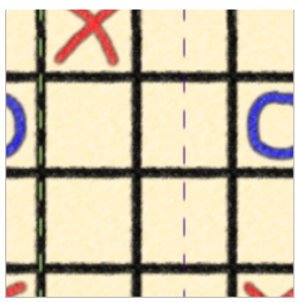
flat Torus:  curved torus: 

flat Klein bottle:  curved Klein bottle: 

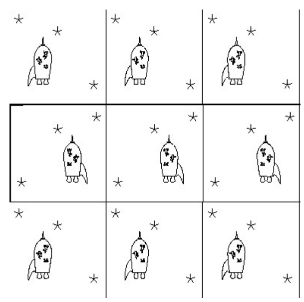
CC-By-2.5 by Tttrung

The curved Klein bottle has a nasty intersection when the slinky passes through itself. Our gluing instructions give no hint of this—the intersections only arose when we tried to glue the space in three dimensions, picking up distortions. In the fourth dimension there is enough room to glue the edges together without creating intersections, and it is here where the glued square really exists as a flat Klein bottle.

If we sketch what a creature living on the flat Klein bottle would see in all directions, this is a tiling view, the large-scale/global view. The top left is just below the bottom right in this Klein bottle universe because they are identified by the mirror reflection. However, if we look off to the right or left, we see exactly the same image instead of the reflected one:



Jeff Weeks, Torus Games



Heidi Burgiel, The Geometry Center

The tiling views show a flat Klein bottle that satisfies the laws of Euclidean geometry from high school because we can tile the plane with squares without distortion–four 90 degree angles come together at a vertex to form 360 degrees.

Experience what it is like to live on a Klein Bottle by downloading the Torus Games, free software that will work on a computer, tablet, or phone, from the 2D universes hand in link, and playing Klein Bottle Tic-Tac-Toe:

- Select **Tic-Tac-Toe**.
- Using the Topology menu, change to **Klein Bottle**.
- You are allowed to scroll the board by clicking on a grid line, holding down and moving. Move the hand that results to the right and see like it behaves like a Pac-Man. Move up to the top and see the reflection about the vertical purple dashed line. **On a computer, the esc key will let you access the menus.**
- The Rules: When the purple dotted line of reflection (the one through the middle of the squares) is placed in the middle of the screen, then the top left square is just below (out of view) the bottom right square in this Klein Bottle universe. The yellow dotted lines show the straight across identifications left to right, as you can see when you scroll horizontally.

- Erase Tic-Tac-Toe Board will bring up a new game.

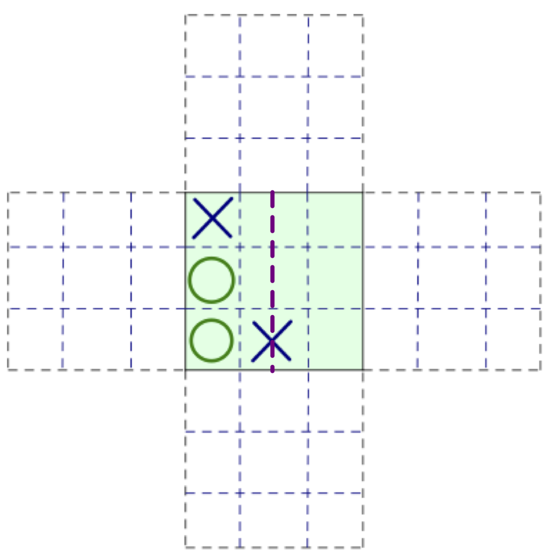
Be sure you are in a Klein bottle and set the purple dotted line of reflection in the middle. Play and identify the winner where you scroll the board while playing to get a sense of the tiling—the global view—best out of 3 (your response can be who won and/or some pictures if you would find them helpful).

16. Next scroll the board so that the purple dotted line of reflection is in the middle of the board. Leave it there.

    Play and identify the winner where you DO NOT scroll the board (local view)—best out of 3:

17. Play enough times of this, or another one of the games (under Change Game), that you understand the visualization of the Klein bottle tiling. The Klein bottle Pool Balls or the Klein bottle Maze are especially nice visualizations.

18. Next fill in the following—sketch a global tiling view of the board, where left and right is the same as the board, but above and below are reflected (with the left and right columns swapped). If you are stuck, play some more games.



19. Where can x go to win on the main board on the very next move? Mark a winning move there, only in the center board.