universe lab

Dr. Sarah's 1010: Introduction to Mathematics: Geometry of the Earth and Universe

How we measure and view the world around us and decide what is the nature of reality.

goals:

- Develop problem solving and analysis skills in recognizing patterns and similarities in geometric representations to work towards becoming logical, flexible, critical thinkers and problem solvers.
- Explore applications of geometry in everyday life.
- Employ spreadsheets and geometry to apply numerical representations in tables to the real-world.
- Compare and contrast small-scale and large-scale mathematical regions.
- Communicate geometric information in written documents.

You'll see connections to art, philosophy, physics, medicine, astronomy, and visualization. Because our brains are wired to see 3D (by using layered 2D slices!), if you are properly engaging the material with an open mind, then these ideas should stretch the limits of your imagination. In order to help you, I have pulled together a variety of activities.

Real-life Applications of Related Material

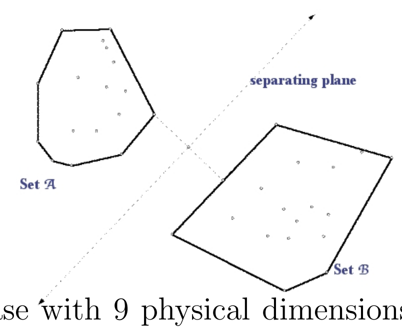
1. There are many real-life applications of the geometric spaces we have been studying, such as:

   higher dimensional hyphersphere: artificial intelligence and machine learning, Einstein's theory of relativity, protein modeling in biology, statistics

   hyperbolic geometry: hyperbolic and crystal structures can store more hydrogen or absorb more toxic metals, artists and mathematicians have been creating hyperbolic coral reefs, hyperbolic maps of the brain to diagnose or monitor neurological diseases, hyperbolic maps of the internet to try and reduce the loads on routers, Mercury's orbit about the Sun is slightly more accurately predicted when hyperbolic geometry is used in place of flat Euclidean geometry

   Download the data file **heartdata.xlsx**, which is available in the assignment on ASULearn, and open it in Excel. The complexity of higher dimensions can be experienced regularly in our data driven society. Any time we measure more than 3 variables for a poll, we are inside of a higher dimensional space. Each column is a different dimensions worth of data. How many dimensions is this space (i.e. how many columns)?

2. This data was collected from patients in the US, Hungary and Switzerland. Each patient is a different row. How many patients were studied? Subtract 1 from the rows used, since the first row contains the data descriptors rather than a person.



Each individual corresponds to a point in an n-dimensional space where n is the number of measurements recorded for each individual. Mathematics is then used to separate the classes via a plane, somewhat similar to the idea of linear regression (which we'll see later), but instead of finding a "best fit" line to all of the data, we find the higher dimensional plane that best separates the data into classes.

The same method was used for the Wisconsin Breast Cancer Database with 9 physical dimensions. New individuals are then classified and diagnosed by a computer using the separating plane. There has been 100% correctness on computer diagnosis of 131 new (initially unknown) cases, so higher dimensions can save lives!

Models for the Shape of the Universe

NASA states that "the universe is flat with only a 0.4% margin of error," so all geometries are possible: "we can only observe a finite volume of the Universe. All we can truly conclude is that the Universe is much larger than we can directly observe." One possibility is an infinite Euclidean space. Another is a Euclidean wraparound space, a finite Euclidean cubical block of space with faces glued together.

Euclidean wraparound spaces and their gluings

3. Open up **Torus Games**, which you also used in the 2D universes hand in assignment, and can be found in today's lab on ASULearn too. Select **3D Maze**.

   Drag the sphere through a face. Don't worry about finishing the maze, but do play enough so that you get the idea of the gluing. You can also rotate the figure to see the maze better. This is one of the possibilities for a finite universe without any edges (since we glue them together), and many 3-D video games (like Valve's Portal) also make use of these types of universes.

   This is a 3-torus, where the corresponding points are glued directly across from each, i.e. connected to each other, rather than glued over twists and turns—a reflection or rotation. As you visualize this, circle each gluing:

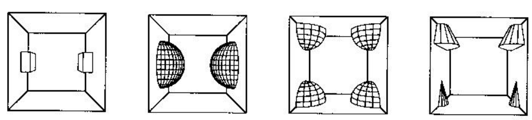
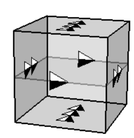
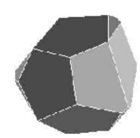
   circle: directly across   for the front to back face gluing once you've internalized it

   circle: directly across   for the left to right face gluing once you've internalized it

   circle: directly across   for the top to bottom face gluing once you've internalized it

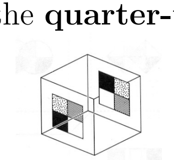
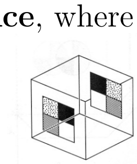
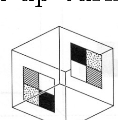
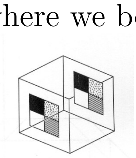
4. Each Euclidean 3-torus below has a familiar surface drawn in it once you perform the gluings of faces directly across. For example, the fourth picture is a cone because the top and bottom face gluing yields 2 half cones, and then the left and right faces are glued together forming 1 full cone. The second and third generates the same figure! List the 2 names to fill in the blanks of the surfaces in the 3-torus below as you internalize the visualization—choose from **cube** or **sphere**:

   Jeff Weeks: *Exploring the Shape of Space*

   ________   ____________________   cone   UIUC: 3-torus gluings   Weeks manifold gluings

5. Below, unmarked walls are glued to one another in the simple, straight-across way while the marked side shows whether to glue straight across, with a reflection or a rotation by identifying corresponding squares (squares that are filled in the same get glued together). For example, the second space is the Klein space because it is identified via a vertical mirror plane of reflection that cuts the cube in half. The half-turn space is a 180° rotation of the front to get to the back. For the remaining two spaces, fill in the blanks: One is the **3-torus**, with identifications straight across, and the other is the **quarter-turn space**, where we end up turned 90° from where we began as we fly across:

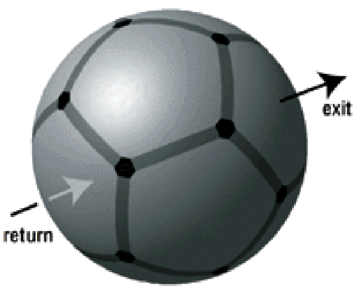
   Jeff Weeks: *Exploring the Shape of Space*

   __________________   Klein space   half-turn space   __________________

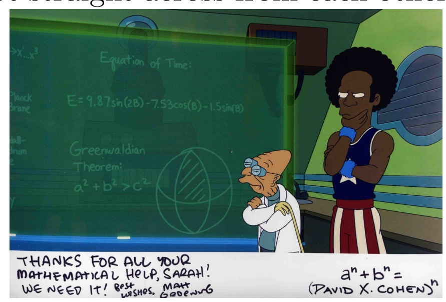
   If our universe is one of these we might be able to tell by looking for repeated patterns of stars.

Spherical wraparound spaces

Historically, Aristotle argued that the universe is finite on the grounds that a boundary was necessary to fix an absolute reference frame, which was important to his worldview. But his critics wondered what happened at the edge. German mathematician Georg Riemann solved the riddle in the mid-19th century. As a model for the cosmos, he proposed the hypersphere. It was the first example of a space that is finite yet has no problematic edges to fall off of. One might still ask what is outside the universe. but this question supposes that the ultimate physical reality must be that the space sits inside of something else. Nature, however, need not cling to this notion. It would be perfectly acceptable for the universe to be a hypersphere and not be embedded in any higher-dimensional space. Such an object may be difficult to visualize, because we are used to viewing shapes from the outside. But there need not be an "outside." If we want to glue opposite faces together to form a wraparound spherical dodecahedral space, we will need to do so with a twist, a rotation of 36°, since opposite faces are not straight across from each other.







Jean-Pierre Luminet: shape of space   Paul Nylander: life from the inside   Bender's Big Score: Greenwaldian thm

6. Access the **dodecahedron movie** from the universe lab on ASULearn. How many faces does a dodecahedron have?

7. Which of the following is true:
   Did the flat faces of 3 dodecahedron fit together perfectly in the movie, leaving no space in excess?
   —or—
   Is there extra space that can be filled if we bow the faces out to create a spherical space?

Hyperbolic wraparound space

8. We can glue together corresponding sides of a bowed-in hyperbolic figure with 18 faces. Just as Pythagoras was not the first to explore with the Pythagorean theorem [which goes back to Babylonian times], I was not the first to come up with the spherical version now known as the Greenwaldian theorem in the *Futurama* universe [which probably goes back to Menelaus of Alexandria]. However, the Weeks manifold, a wraparound space named after Jeff Weeks, was first discovered by him. The gluing instructions are in a picture within the 3-torus section above, but the faces would be bowed-in. There are infinitely many possible topologies for a finite wraparound hyperbolic space and their rich structure is still the subject of intense research. Search the web for
   Weeks manifold
   Write down one related item.

What is the 4th Physical Dimension?

The current scientific consensus is that the universe has more physical dimensions than the three we directly experience. In *Hyperspace & A Theory of Everything*, Dr. Michio Kaku says:

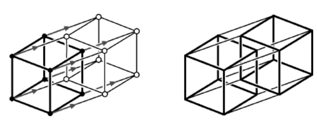
> When I was a child, I used to visit the Japanese Tea Garden... The carp would only be able to swim forwards and backwards, and left and right. But I imagined that the concept of "up", beyond the lily pads, would be totally alien... One day it rained, and I saw the rain drops forming gentle ripples. The third dimension would be invisible to them, but vibrations in the third dimensions would be clearly visible. These ripples might even be felt by the carp,

> who would invent a silly concept to describe this, called "force." They might even give these "forces" cute names, such as light and gravity. We would laugh at them, because, of course, we know there is no "force" at all, just the rippling of the water. Today, many physicists believe that we are the carp swimming in our tiny pond, blissfully unaware of invisible, unseen universes hovering just above us in hyperspace. We spend our life in three spatial dimensions, confident that what we can see with our telescopes is all there is, ignorant of the possibility of 10 dimensional hyperspace. Although these higher dimensions are invisible, their "ripples" can clearly be seen and felt. We call these ripples gravity and light.

We can try and understand the 4th physical dimension by thinking about how a carp, 2D Marge Simpson or Arthur Square can understand the third dimension. For example, when they experience 2D slices of global 3D behavior, it is in this indirect way that they can gain an appreciation for 3D. Similarly, we can use local 3D views/slices of higher dimensions to try and gain an appreciation for the counterintuitive behavior of global 4D objects.

9. What might one layer of Homer's skin look like in 4D if he were to change from 3D to 4D—a layer of skin looks like a 2D piece of paper with holes or pores in it—think about what familiar dairy product—what kind of cheese named after a country has holes—this might resemble if that thin layer gained a dimension/i.e. gained thickness.

10. How is a hypercube formed from a cube? Fill in the blanks of the analogy where Frink is now talking about 4D (and notice that it is similar to Professor Frink's description of how a square can be extended to form a cube from the 2D universes hand in). Use the images to help:

    Into the Fourth Dimension by Davide P. Cervone

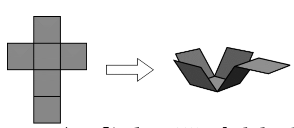
    Frink: suppose we exte-end the 3D cube beyond the 3 dimensions of our universe (along the hypothetical axis, there), moving perpendicular to itself to sweep out a figure.

    Everyone: [gasps]

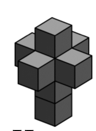
    Frink: This forms a ______dimensional object known as a ______________, or a "Frinkahedron"

    We don't have access to the perpendicular direction, so we can experience only a distorted figure.

11. A different visualization is to look at gluing instructions, like we did for a cube. However, we can't see what the hypercube folds up into, because we don't have the fourth dimension.

    Davide P. Cervone's Cube Unfolded

    Hypercube Unfolded

    Search the web to find what famous artist represented the unfolded hypercube in his 1954 painting?

12. Examine **The Cubical Faces of a Hypercube movie** by Davide P. Cervone, which is available in the universe lab on ASULearn. A hypercube is one of the first 4D objects many try to understand. Yet, physicists and mathematicians assert that it cannot be the shape of our universe since it has an edge, which means that there would have to be something on the other side of the edge. Notice that the hypercube is put together visually in ways we can't fully process because we would need the full use of four dimensions to properly see it, just like Arthur Square or 2D Marge trying to grasp

3D objects. The two regular cubes are the "bottom" and "top"—the initial and final positions of the cube in motion. The others are distorted in our view, but are regular cubes in 4D, and they are created by the interconnections in 4D that are swept out.

Count the total number of cubes that are boundary "faces" of the hypercube in the movie—they are highlighted in the movie and are distorted in our view but there is enough space for them to be cubes in higher dimensions. How many are there?

13. Look for a pattern between the dimension and the number of boundary pieces. Use your response from the last question to fill in the hypercube row, and then figure out the n-dimensional version.

figure name	dimension of object	number of boundary pieces	type of boundary pieces
point	0D	0=2(0)	impossible
line	1D	2=2(1)	points as boundary
square	2D	4=2(2)	lines as edges
cube	3D	6=2(3)	square faces
hypercube	4D	_____	cube faces
n-cube	nD	_____ leave n general Hint: function of n	$(n-1)$-cube

14. Artist Tony Robbin, is another artist who creates shadows of higher dimensions on canvas:

I find it a privilege to live in a time where advanced mathematics is represented in pictures instead of only equations. [Tony Robbin, *Seeing is Believing*]

⋆ Search the web for:
Tony Robbin hypercube
and write down one related item
(not to Robbins who may also come up!).

Oil on Canvas, 56 by 70 in

Jeff Weeks Motivations

15. Read the following interview. Circle or write down at least one or two items that you found interesting, disagreed with, had a question on, or wished had been done as related to the following:

What influences led you to become a mathematician?

Reading Abbot's *Flatland* during senior year in high school was a turning point. After two weeks of intense effort, almost in a flash I could finally "see" it. This glimpse into a new world made a huge impression on me. I think that's what hooked me on mathematics, and on geometry in particular. I love the connections between pure mathematics and the physical world. A real joy.

Why did you become a mathematician?

The combination of beauty and simplicity of exploring new worlds exceeds anything you'll find in everyday life. As part of the current research I've been working on the hypersphere. That's part of the joy of mathematics—you might struggle with something for months, but then when you "get it"...

Are there any local to global issues or diversity issues in your experiences?

In an international sense mathematics is extremely diverse, with people from a broad range of cultures sharing a common mathematical culture. Domestically, though, mathematics seems to be in decline, regardless of race or gender. The US now relies overwhelmingly on foreigners both for mathematically skilled positions in industry and for graduate students in pure math. I did once serve as a "de facto mentor" for a woman's undergraduate thesis. She did a super job, went on to graduate school, and is now a tenured math professor. I don't really see this as a "diversity issue," though. She did the same project a male student would have done. So in that sense mathematics is pretty gender-neutral. It's the social environment where things get messy, but that seems to vary enormously, from departments where gender isn't an issue to other departments that openly discourage women students and faculty.

Did you have support from family and society or face any barriers?

I don't think my parents ever grasped why I wanted to go into math (even when I was applying to graduate school my dad sat down with me and earnestly and kindly suggested that I might want to consider business school as an alternative). On the other hand, my parents fully supported my decisions and encouraged me at every step of the way. Society too has been supportive. However, in graduate school, the pressure took all the joy out of it, and under those circumstances it was hard to make progress. I didn't start enjoying mathematical research until after I got my PhD. I've taken a non-standard path, but there's always been a way to make things work. [After several years of teaching undergraduate mathematics, he resigned to care for his newborn son, and continued his research as a freelance mathematician.] This is a great time to be a mathematician.

Describe the process of doing mathematics?

Everything I do I see as pictures. The mental images are totally convincing. In cosmic topology, the problems arise from anomalies in our observations of the universe. I like to start with the absolutely simplest case that isn't totally trivial. If it's too messy I'll spend a few days trying to understand more deeply what's going on, in hopes of finding a simpler proof or a more enlightening point of view. This is particularly true of calculations—a calculation can prove that something is true without telling you *why* it's true.

How do you get the flashes of insight that you need to do research?

If I could tell you that, I'd be a far more productive mathematician :-) Seriously, for me the key is to be well-rested, unrushed and undistracted. Math is discovered, not invented. For me there is no doubt. As you explore, you work your way through a lot of "false understandings" where things don't quite come together. Then, all of a sudden, things start falling into place. That's the moment you realize you're onto something. It's very much an experience of discovering something that's already there. I find that a lot of progress takes place in my subconscious. That is, I can go to bed totally confused about a question and wake up with an idea. Similarly, I might be, say, out for a bike ride and find an idea just pops into my head, without my having been aware that I was even thinking about the problem. (Your students should understand, though, that they can't expect an idea to literally come from nowhere—they have to immerse themselves in the question first!)

References

Adapted from excerpts taken from:
Davide P. Cervone's materials
Cathy Gorini *Geometry at Work*
David Henderson *Experiencing Geometry in the Euclidean, Spherical, and Hyperbolic Spaces*
Jean-Pierre Luminet, Glenn D. Starkman and Jeffrey R. Weeks *Is Space Finite?*
Jeff Weeks *Exploring the Shape of Space*
my Jeff Weeks Interview