# 10.2 Taylor Series Group Work Target Practice

Identify this as a Taylor series of a known function and sub for $x$ to find the sum of $1-\dfrac{3^2}{2!}+\dfrac{3^4}{4!}+\ldots$.

1) Which is it? $e^3$, $\cos(3)$, $\sin(3)$, or $\frac{1}{1-3}$

2) What do the other Taylor series look like?

b) First, note that $1-\dfrac{3^2}{2!}+\dfrac{3^4}{4!}+\ldots$ is an alternating series that has even powers of 3, which fits the even function $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}\ldots = \displaystyle\sum_{i=0}^{\infty}(-1)^i\frac{x^{2i}}{(2i)!}$. When we substitute x=3, we see that it identically matches, so this sum is $\cos(3)=\displaystyle\sum_{i=0}^{\infty}(-1)^i\frac{3^{2i}}{(2i)!}$, or equivalently $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}}{(2n)!}$ (as $i$ and $n$ are both common in various contexts).

For the other Taylor series, plug in 3 for x:

a) $e^x = 1+x+\frac{x^2}{2!}+\ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $e^3 = 1+3+\frac{3^2}{2!}+\ldots = \sum_{n=0}^{\infty}\frac{3^n}{n!}$

c) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ so $\sin(3) = 3-\frac{3^3}{3!}+\frac{3^5}{5!}\ldots = \sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}}{(2n+1)!}$

d) $\frac{1}{1-x} = 1+x+x^2+\ldots = \sum_{n=0}^{\infty}x^n$ so $\frac{1}{1-3} = 1+3+3^2+\ldots = \sum_{n=0}^{\infty}3^n$

The first 3 have infinite radius of convergence and the last 2 have $r=1$.

Compute the Taylor series for $f(x)=\ln x$ at the point $a=1$.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(a)$ | Taylor Term $\frac{f^{(n)}(a)}{n!}(x-a)^n$ |
|---|---|---|---|
| 0 | ln(x) | ln(1)=0 | $\frac{0}{0!}(x-1)^0 = 0$ (note $0! = 1$) |
| 1 | $\frac{1}{x}=x^{-1}$ | $\frac{1}{1}=1$ | $\frac{1}{1!}(x-1)^1=(x-1)$ |
| 2 | $-x^{-2}$ | $-1^{-2}=-1$ | $\frac{-1}{2!}(x-1)^2=-\frac{1}{2}(x-1)^2$ |
| 3 | $--2x^{-3}=2x^{-3}$ | $2(1)^{-3}=2$ | $\frac{2}{3!}(x-1)^3=\frac{2}{3\cdot 2!}(x-1)^3=\frac{1}{3\cdot}(x-1)^3$ |
| 4 | $-3\cdot 2x^{-4}$ | $-6(1)^{-4}=-6$ | $\frac{-6}{4!}(x-1)^4=\frac{-6}{4\cdot 3!}(x-1)^4=\frac{-1}{4\cdot}(x-1)^4$ |

Notice the cancellation in the Taylor terms that allows reduction from $n!$ to $n$—this worked because as those powers pulled down that was exactly $(n-1)!$ and $\frac{(n-1)!}{n!}=\frac{(n-1)!}{n(n-1)!}=\frac{1}{n}$

The Taylor polynomial of degree 4 is the sum of the Taylor terms:

$P_4(x)=(x-1)-\dfrac{1}{2}(x-1)^2+\dfrac{1}{3}(x-1)^3-\dfrac{1}{4}(x-1)^4+\ldots$

From here we can see the pattern of the alternating series and the coefficients of $\pm\frac{1}{n}$, so the Taylor series is $\displaystyle\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{n}(x-1)^n$. The first nonzero term, the degree 1 term of $(x-1)$, is positive, so we can see it should be $(-1)^{n+1}$ rather than $(-1)^n$.