1120 - Section 10.3 Finding and Using Taylor Series

Recall the following Taylor series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for all x}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for all x}$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \text{ for all x}$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1$$

Composition 1: Use a known Taylor series to find the Taylor series about 0 for $\sin(2x^3)$ (Hint: substitute i.e/ composition of functions).

We can substitute $2x^3$ for x in $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

$\sin(2x^3) = 2x^3 - \frac{(2x^3)^3}{3!} + \frac{(2x^3)^5}{5!} - \frac{(2x^3)^7}{7!} + \cdots = 2x^3 - \frac{8x^9}{3!} + \frac{32x^{15}}{5!} - \frac{128x^{21}}{7!}$ for all x.

Composition 2: Use a known Taylor series to find the Taylor series about 0 for $\frac{1}{1+x^2}$.
(Hint: substitute $-x^2$ for x)

We can substitute $-x^2$ for x in $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.

$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 = 1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ for } |x| < 1$$

The radius of convergence is inherited from the original series, so what is the radius of convergence?

The radius of convergence is 1 since it was inherited from the geometric series with the fixed ratio of x and starting term of 1.

Differentiation: We can take derivatives of Taylor series: just take the derivative term by term. The radius of convergence of the derivative will be the same as that of the original series. To see this, take the derivative of the Taylor series of the sine function, term by term, to get the Taylor series of the cosine function. Reduce to show that you get the usual Taylor series of cosine.

$$\cos x = \tfrac{d}{dx} \sin x = \tfrac{d}{dx}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Integration: It is also possible to integrate Taylor series term-by-term. Use the Taylor series of $f(x) = \frac{1}{1+x^2}$ we did this in Composition 2 above) to find the Taylor series of $\arctan x$. What is the radius of convergence?

The radius of convergence is inherited from the parent series:

$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 = 1 - x^2 + x^4 - x^6 + x^8 + \cdots \text{ for } |x| < 1$$

Integrate term by term: $\arctan x = \int \frac{1}{1+x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

Estimating an Integral: We can use some of the terms to estimate a number like π, but we can also integrate Taylor terms to estimate an integral.
Part a) Find the first three terms of the Taylor series for $\sin(x^2)$ then
Part b) Find the first three terms of the Taylor series for $\int \sin(x^2)\, dx$.
Part c) To find an estimation of $\int_0^1 \sin(x^2)\, dx$, evaluate the response from Part b) at the endpoints of the interval (no need to simplify).

Part a) For $\sin(x^2)$ we can substitute x^2 for x in $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.
So the first three terms are $\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$

Part b) Now integrate term by term: $\int \sin(x^2)dx$ is approximated by the integral of the first three terms: $\int x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} dx = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!}$

Part c) Find an estimate of $\int_0^1 \sin(x^2)\, dx$ by plugging in 0 and 1: $= \frac{1^3}{3} - \frac{1^7}{7 \cdot 3!} + \frac{1^{11}}{11 \cdot 5!} - \frac{0^3}{3} - \frac{0^7}{7 \cdot 3!} + \frac{0^{11}}{11 \cdot 5!}$

Product: We can also find the Taylor series of a product of two Taylor-expandable functions. Apply this idea for the function $h(x) = e^x \cos x$. Find the degree 3 Taylor polynomial.

We will multiply the two Taylor series:
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for all x
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x

$e^x \cos x = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots)(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots)$

The degree 3 polynomial will be all terms where the power of x is less than or equal to 3. We foil to find them:
$= 1 \cdot 1 \quad + 1 \cdot \frac{-x^2}{2!} \quad + x \cdot 1 \quad + x \cdot \frac{-x^2}{2!} \quad + \frac{x^2}{2!} \cdot 1 \quad + \frac{x^3}{3!} \cdot 1$

Now combine like terms
$= 1 \quad + x \quad - \frac{x^2}{2!} \cdot 1 + \frac{x^2}{2!} \cdot 1 \quad + x \cdot \frac{-x^2}{2!} + \frac{x^3}{3!} \cdot 1$
$= 1 + x + 0 - \frac{x^3}{2} + \frac{x^3}{6}$
$= 1 + x - \frac{x^3}{3}$