10.4 Error Bounds Solutions

1. Use the Lagrange Error Bound/Taylor's Inequality to find the maximum possible error in estimating e^{-2} using the Taylor polynomial of degree 8 about 0 for $f(x) = e^x$.

   The Lagrange Error Bound/Taylor's Inequality is written as $|f(x) - P_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$, with $|f^{(n+1)}| \leq M$ for a to x. Plug in $n = 8$: $|f(x) - P_8(x)| \leq \dfrac{M}{(8+1)!}|x-a|^{8+1} = \dfrac{M}{(9)!}|x-a|^9$, with $|f^{(9)}| \leq M$ for a to x.

   Here a is the center, 0, where there is no error, and x is how far out we are going from the center, in this case to -2. So we need the 9th derivative of e^x to find M. Since $\frac{d}{dx}e^x = e^x$, each derivative is the same. To find M we need to bound the absolute value of the 9th derivative on the interval from the center 0 to -2. e^x, is an increasing function, so it attains its maximum on the right most point, which is 0 in this case. Thus the maximum is $e^0 = 1 = M$.

   Hence the error bound in using $P_8(x)$ to approximate e^x here is $\dfrac{1}{(9)!}|-2-0|^9 = \dfrac{2^9}{9!} \approx .0014$

2. Compute the linear approximation of $\arctan x$ about 0. Then use the Lagrange Error Bound/Taylor's Inequality to find the maximum possible error in using it to estimate $\arctan \frac{1}{2}$.

   First we compute the degree 1 Taylor polynomial about $a = 0$:

n	$f^{(n)}(x)$	$f^{(n)}(a)$	Taylor Term $\dfrac{f^{(n)}(a)}{n!}(x-a)^n$
0	$\arctan x$	$\arctan 0 = 0$	$\dfrac{0}{0!}(x-0)^0 = 0$ (note $0! = 1$)
1	$\dfrac{1}{1+x^2}$	$\dfrac{1}{1+0^2} = 1$	$\dfrac{1}{1!}(x-0)^1 = x$

   We add the Taylor terms: $P_1(x) = 0 + x = x$.

   Next we'll use the Lagrange Error Bound/Taylor's Inequality to find the maximum possible error in using $P_1(x) = x$ to estimate $\arctan \dfrac{1}{2}$.

   Plug in $n = 1$: $|f(x) - P_1(x)| \leq \dfrac{M}{(1+1)!}|x-a|^{1+1} = \dfrac{M}{(2)!}|x-a|^2$, with $|f^{(2)}| \leq M$ for a to x.

   Here a is the center, 0, where there is no error, and x is how far out we are going from the center, in this case to $\frac{1}{2}$. So we need the 2nd derivative of $\arctan x$ to find M. We already computed the first derivative $\dfrac{1}{1+x^2} = (1+x^2)^-1$. Apply power rule and chain rule to take the derivative: $f^{(2)}(x) = -(1+x^2)^{-2}2x = \dfrac{-2x}{(1+x^2)^2}$

   To find M we need to bound the absolute value of the 2nd derivative on the interval from the center 0 to $\frac{1}{2}$. If we plug 0 in to $|f^{(2)}(x)| = |\dfrac{-2x}{(1+x^2)^2}|$ we have 0, and if we plug $\dfrac{1}{2}$ in we have $\dfrac{2\frac{1}{2}}{(1+\frac{1}{2}^2)^2}$. The absolute value of the derivative increases over that interval, so the bound is achieved at the rightmost endpoint. Thus the maximum is $M = \dfrac{2\frac{1}{2}}{(1+\frac{1}{2}^2)^2}$.

   Hence the error bound in using x to approximate $\arctan x$ here is $\dfrac{\frac{2\frac{1}{2}}{(1+\frac{1}{4})^2}}{2!}|0-\dfrac{1}{2}|^2 = \dfrac{\frac{2\frac{1}{2}}{(1+\frac{1}{4})^2}}{2!}(\dfrac{1}{2})^2 = .08$