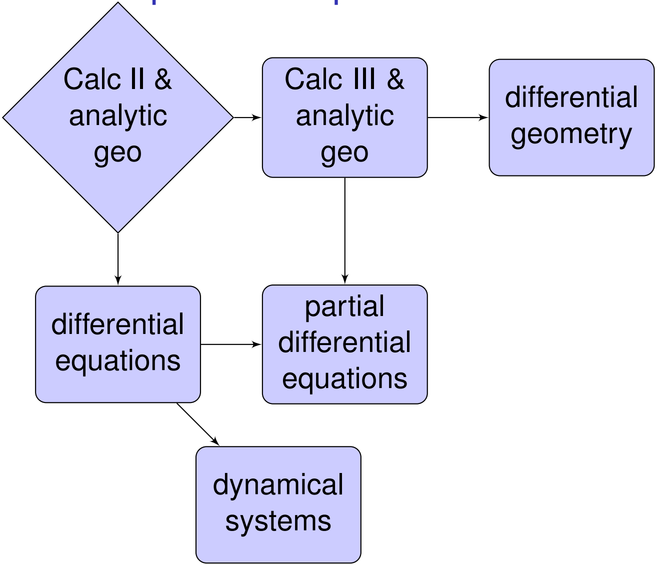
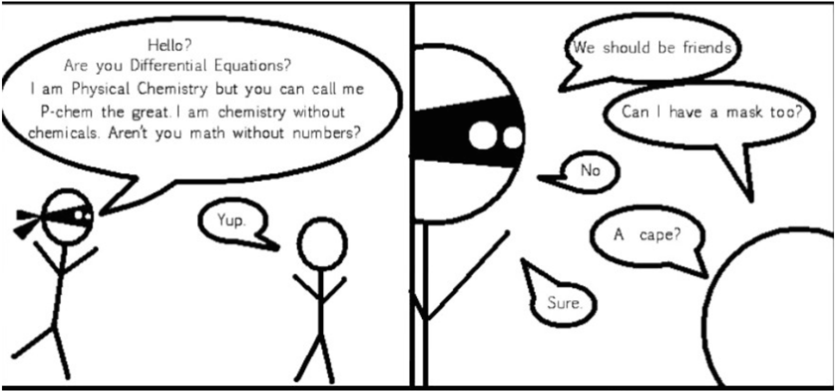
Chapter 11: Equations with Derivatives



11.1: Differential Equations

$$f'(x) = 0 \qquad \frac{dy}{dx} = 0 \qquad y' = 0$$

existence of solutions? uniqueness?



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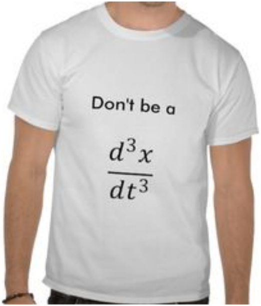

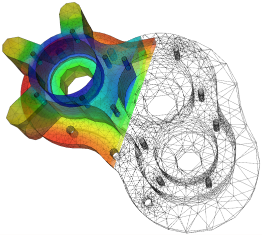

- chemical reactions
- conduction of heat
- continuously compound interest
- growth and decay
- motion of projectiles, planets
- ...

Science is a Differential Equation. Religion is a boundary condition. [Alan Turing]

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Math Electrician

Mark Volt moonlights as a math tutor...

L. Friedman #83 4-26-13
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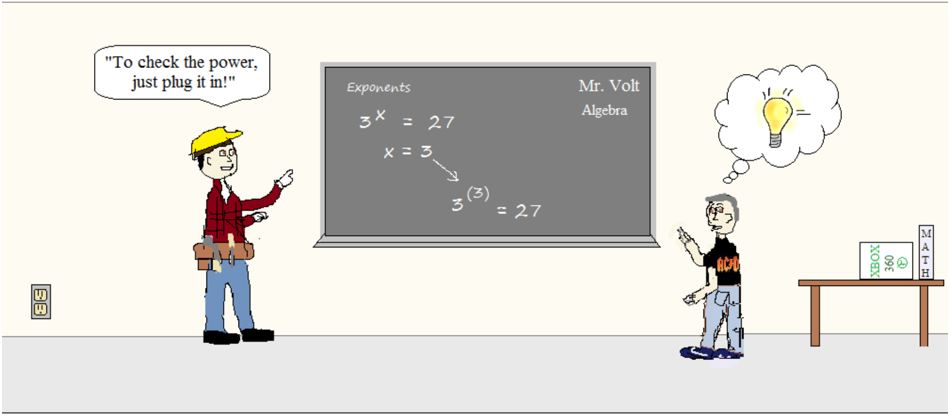

Alex gets the connection!

1. Show $y = \frac{x^2}{2} + c$ is a family of parabolas satisfying $\frac{dy}{dx} = x$.

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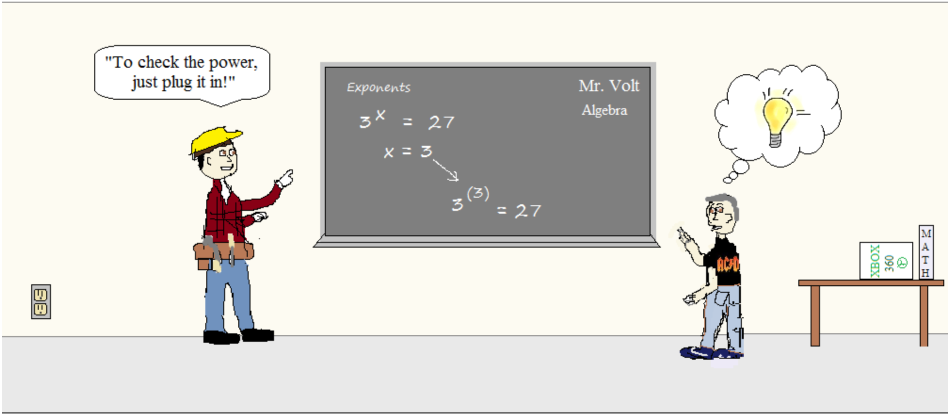

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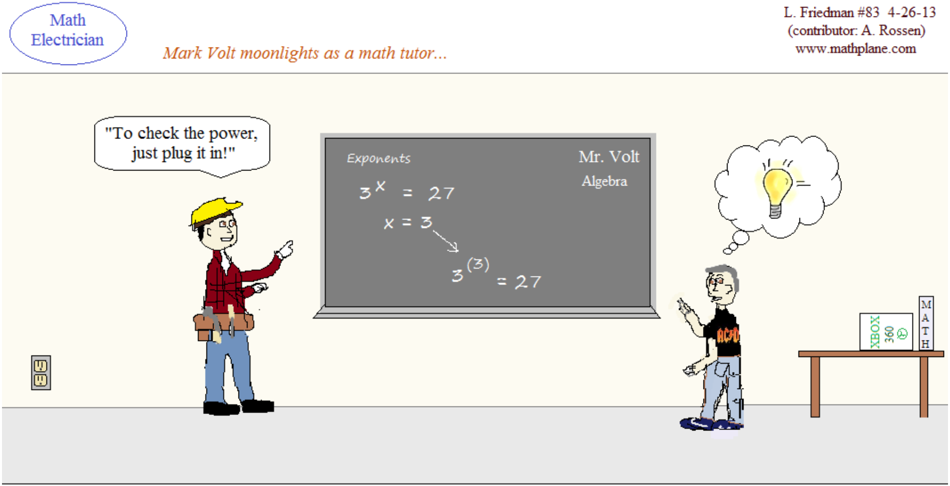
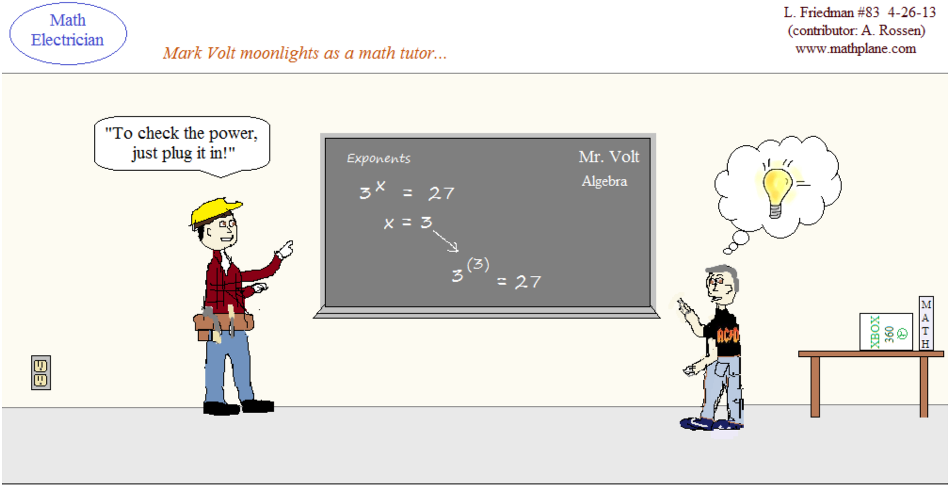
1. Show $y = \frac{x^2}{2} + c$ is a family of parabolas satisfying $\frac{dy}{dx} = x$. Can a solution pass through (2, 5)?



Alex gets the connection!

1. Show $y = \frac{x^2}{2} + c$ is a family of parabolas satisfying $\frac{dy}{dx} = x$.
Can a solution pass through (2, 5)? called an initial condition



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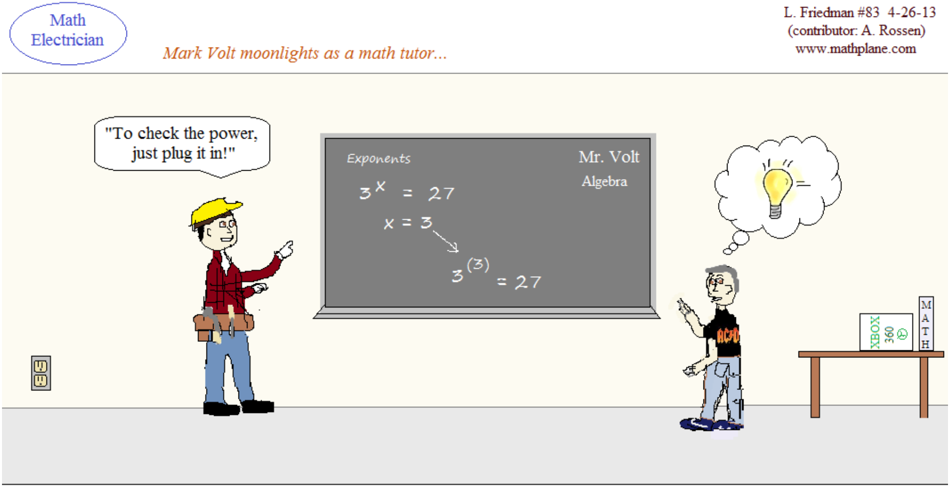
1. Show $y = \frac{x^2}{2} + c$ is a family of parabolas satisfying $\frac{dy}{dx} = x$. Can a solution pass through (2, 5)? called an initial condition
2. Verify $y = \cos \omega t$ satisfies $\frac{d^2y}{dt^2} + 9y = 0$ for 2 possible ω.



Alex gets the connection!

1. Show $y = \frac{x^2}{2} + c$ is a family of parabolas satisfying $\frac{dy}{dx} = x$.
   Can a solution pass through (2, 5)? called an initial condition
2. Verify $y = \cos \omega t$ satisfies $\frac{d^2 y}{dt^2} + 9y = 0$ for 2 possible ω.
3. Can we find k so $y = 5 + 3e^{kx}$ is a solution of $y' = 10 - 2y$?

Clicker Questions

Which of the following is a solution to $y' = ky$?

a) $y(t) = e^{kt}$
b) $y(t) = e^{kt+5} = e^{kt}e^{5}$
c) $y(t) = 2e^{kt}$
d) all are solutions
e) none are solutions

Clicker Questions

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c) $y(t) = 2e^{kt}$
d) all are solutions
e) none are solutions

Which of the following is a solution to $y' = ky$ when $y(0) = 2$?

a) $y(t) = e^{kt+2}$
b) $y(t) = 2e^{kt}$
c) both are solutions with the initial condition
d) none are solutions with the initial condition

Graphical (11.2), Numeric (11.3) & Algebraic (11.4) sols

Isaac Newton used infinite series to solve DEs (1671), like

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

$$x_1 \frac{dy}{dx_1} + x_2 \frac{dy}{dx_2} = y$$

and explore non-uniqueness of solutions.

Gottfried Leibniz introduced the term “differential equations” (1676)

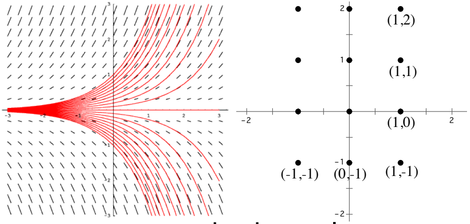
Clicker Question

If $\frac{dy}{dx}$ is 0 at some point, what does that tell you about the tangent line at that point?

a) vertical
b) horizontal
c) makes an angle of 45° with the horizontal
d) undefined

11.2: Slope Fields

Slope field is a set of signposts directing you across the plane.



slope of 0? y=constant: horizontal
infinite slope? x=constant: vertical
slope of 1? y=x: 45° positive slope up to right.
slope of -1? y=-x: $-45°$ negative slope down to right.

11.2: Slope Fields

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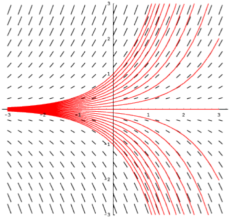

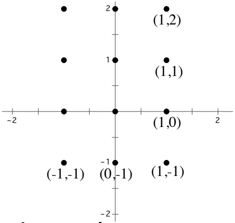

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$1 <$ slope? between 45° and vertical

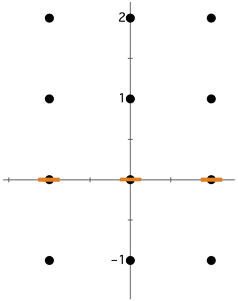
$0 <$ slope < 1? between horizontal and 45°

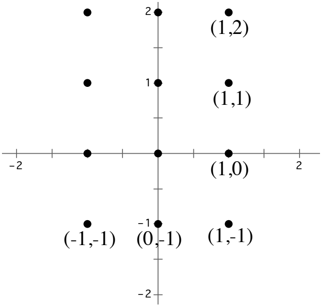
$-1 <$ slope < 0? between horizontal and $-45°$

slope < -1? between $-45°$ and vertical

$$\frac{dy}{dx} = y$$

slope of 0: (-1,0), (0,0) and (1,0). Draw horizontal tick mark.





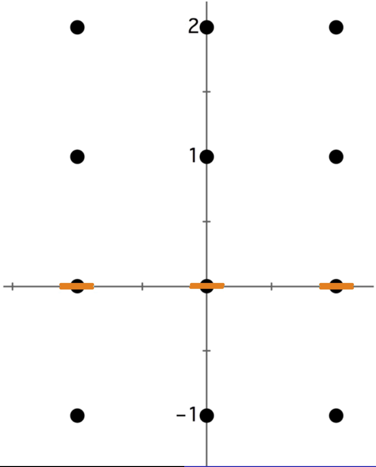
$$\frac{dy}{dx} = y$$

no infinite slope here

no 0 denominators

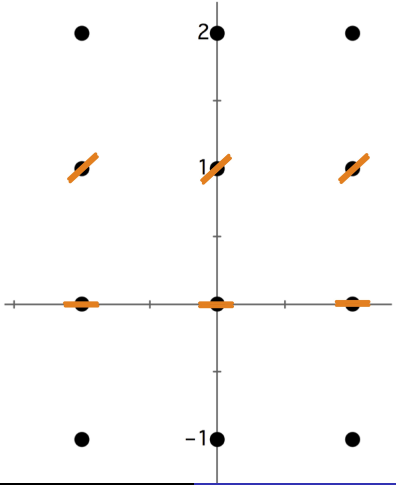
x=constant: vertical

The slope does get more verticle as $|y|$ gets larger



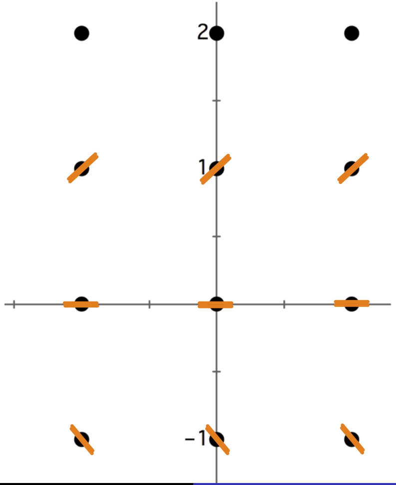
$$\frac{dy}{dx} = y$$

slope of 1: (-1,1), (0,1) and (1,1). Draw 45° positive slope up to right.



$$\frac{dy}{dx} = y$$

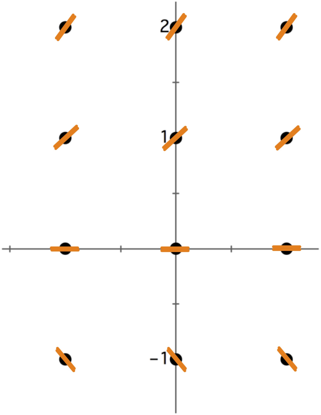
slope of -1: (-1,-1), (0,-1) and (1,-1) Draw 45° negative slope down to right.



$$\frac{dy}{dx} = y$$

What happens at (-1,2), (0,2) and (2,2)? $y' = 2$

+ slope up to right and slightly steeper than 45°



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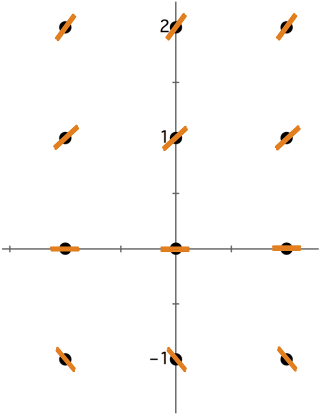
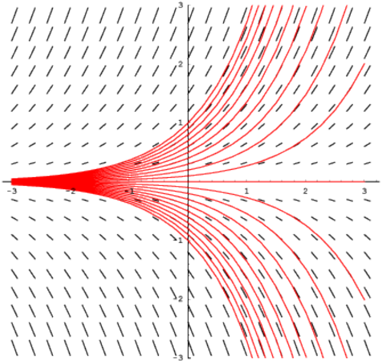
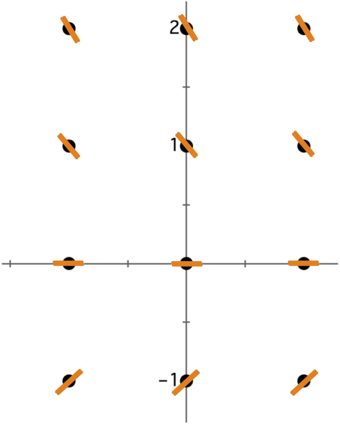
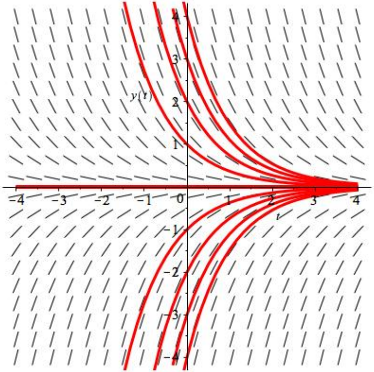




$y = 0$ is an unstable solution

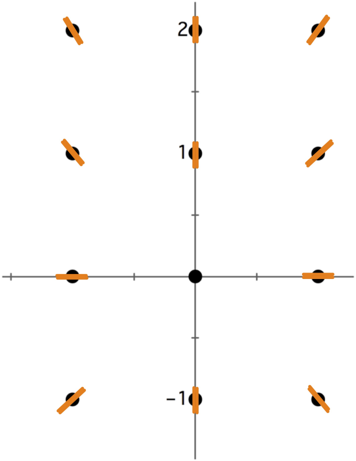
$$\frac{dy}{dx} = -y$$

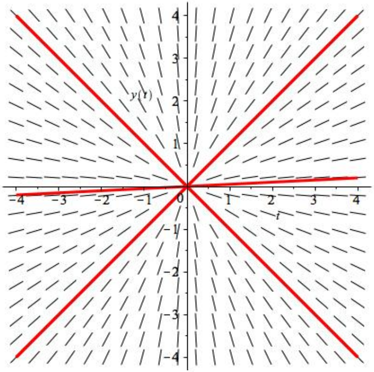




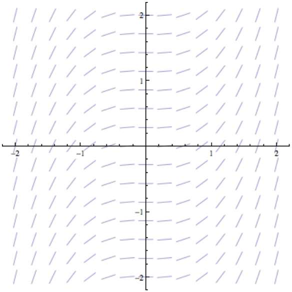
$y = 0$ is a stable solution

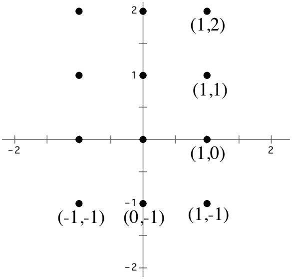
$$\frac{dy}{dx} = \frac{y}{x}$$





Clicker Question





Which differential equation(s) correspond to the slope field?

a) $\frac{dy}{dx} = xy$

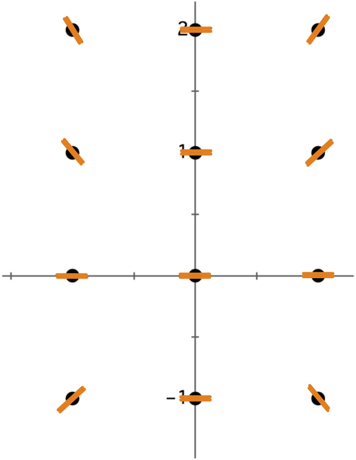
b) $\frac{dy}{dx} = x^2$

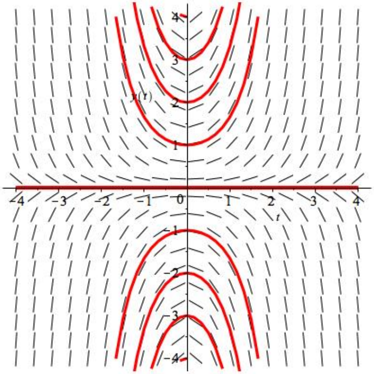
c) more than one of the above

d) none of the above

Slope field is a set of signposts directing you across the plane.

$$\frac{dy}{dx} = xy$$





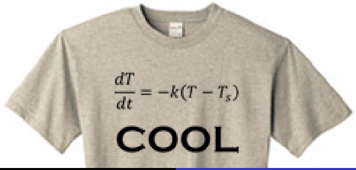
Clicker Question

You pour a cup of coffee at 180° and so Newton's law of cooling applies. Let T be the temperature and t be time.
Then Newton's law of cooling specifies that the differential equation is $\frac{dT}{dt} = -k(T - 72)$, where 72° is the temperature of the room.

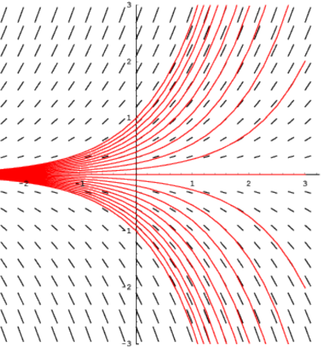
Is $T(t) = 72 + 108e^{-kt}$ a solution to the differential equation?

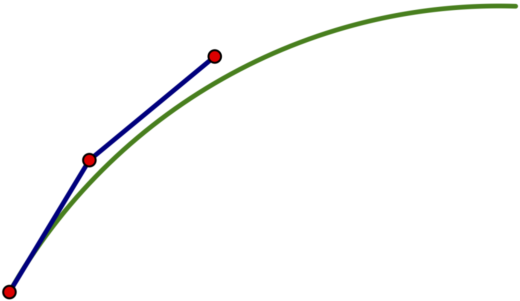
a) Yes and I have a good reason why
b) Yes but I am unsure of why
c) No but I am unsure of why not
d) No and I have a good reason why not

http://www.nerdytshirt.com/cool.html



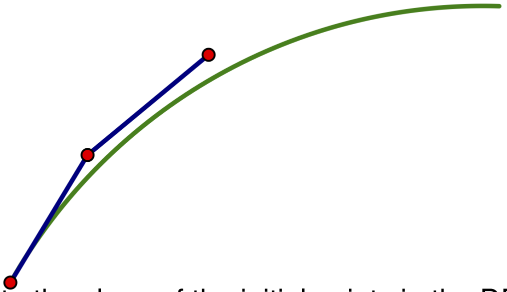
11.3: Euler's Method: Numerical Approx via Slope





11.3: Euler's Method: Numerical Approx via Slope

x $\quad y$ $\quad \frac{dy}{dx}$ $\quad \Delta y = \text{slope} \Delta x$ $\quad (x + \Delta x, y + \Delta y)$



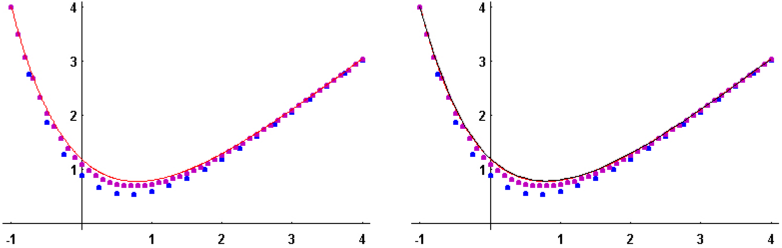
- Calculate the slope of the initial point via the DE
- Head off a small distance Δx (fixed) in that direction to $(x_0 + \Delta x, y_0 + \text{slope } \Delta x)$
- Stop and look at the new signpost— recalculate the slope from the DE, using the new point...
- Example: $\frac{dy}{dx} = x - y$, $\Delta x = .25$, starting at $(-1, 4)$

11.3 Euler's Method: Numerical Approx via Slope

$\frac{dy}{dx} = x - y$ starting at (-1,4). Euler's Method:

x	y	$\frac{dy}{dx}$	$\Delta y =$slopeΔx	$(x + \Delta x, y + \Delta y)$

h=0.25 in blue, h=0.1 in purple, h=0.01 in red, actual solution in black:



http://www.sosmath.com/diffeq/first/numerical/etc/14E4.GIF

Program a smaller time step for better predictions!

https://www.desmos.com/calculator/p7vd3edmei

Clicker Question

x y $\frac{dy}{dx}$ $\Delta y =$slopeΔx $(x + \Delta x, y + \Delta y)$

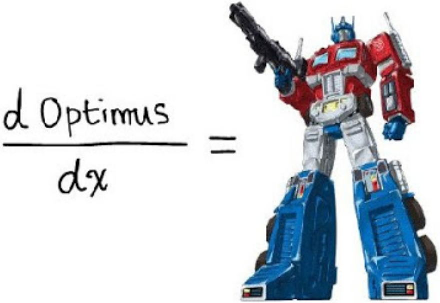
Apply Euler's method one time on

$$\frac{dy}{dx} = (x-2)(y-3)$$

with $\Delta x = .1$, starting at the point $(0, 4)$.

The new point is

a) (.1, 3.9)
b) (.1, 4.1)
c) (.1, 3.8)
d) (.1, 4.2)
e) none of the above



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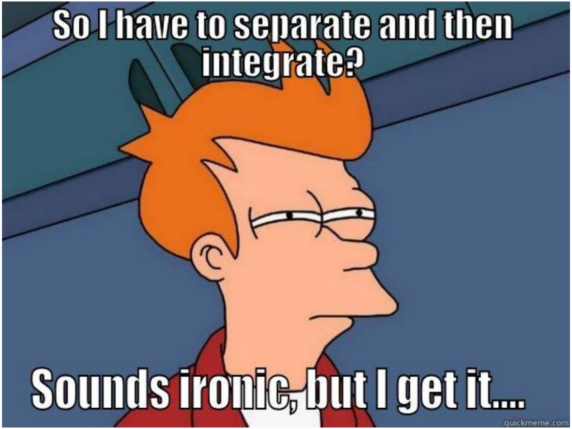
Clicker Question

If a function is decreasing and concave up at (x_0, y_0), what, if anything, can we say about Euler's method $(x_0 + \Delta x, y_0 + \text{slope } \Delta x)$?

a) it will underestimate the true value of the function
b) it will overestimate the true value of the function
c) it will exactly match the true value of the function
d) not enough information is given to be able to determine

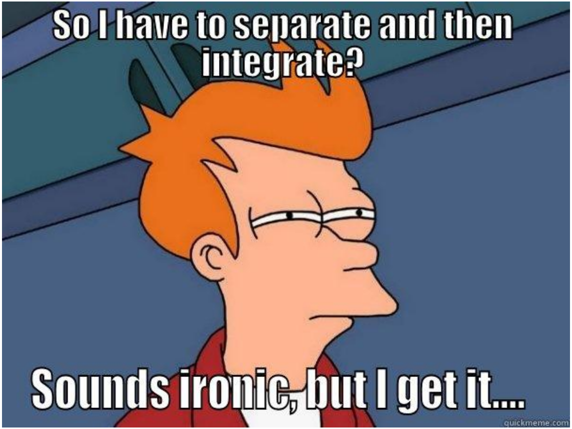
It's called going off on a tangent because it's a derivative of the original conversation [unknown meme author]

11.4: Separable Differential Equations



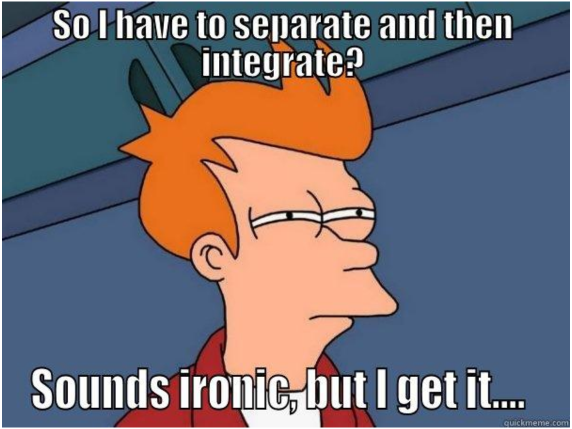
1. $\frac{dy}{dx} = y$

11.4: Separable Differential Equations



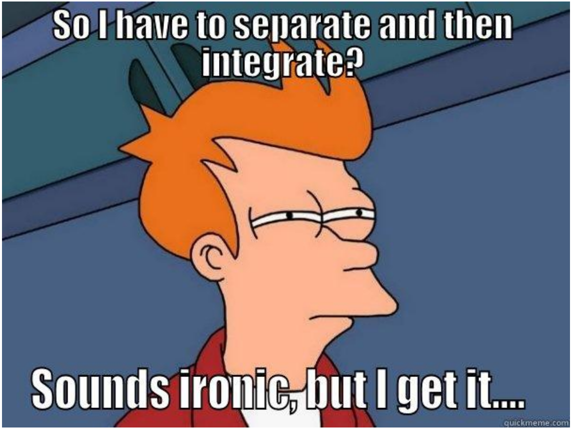
1. $\frac{dy}{dx} = y \qquad y(1) = 3$

11.4: Separable Differential Equations



1. $\frac{dy}{dx} = y \qquad y(1) = 3$
2. $y' = (1 + x)(1 + y^2)$

11.4: Separable Differential Equations



1. $\frac{dy}{dx} = y \qquad y(1) = 3$
2. $y' = (1 + x)(1 + y^2)$

http://www.quickmeme.com/p/3vwufd *Futurama* ™ and © Twentieth Century Fox Film Corporation. Content is not specifically authorized by Twentieth Century Fox.

Clicker Question

If we separate the variables in the differential equation

$$3x\frac{dy}{dx} = y^2$$

we can obtain:

a) $3xdy = y^2dx$

b) $3y^{-2}dy = \frac{dx}{x}$

c) none of the above

Clicker Question

Use separation of variables to find the solution to $\frac{dy}{dx} = \frac{y}{x}$.

a) this differential equation is not separable
b) the solution is algebraically equivalent to $-y^{-2} = -x^{-2} + c$
c) the solution is algebraically equivalent to $|y| = c|x|$
d) none of the above

Clicker Question

Use separation of variables to find the solution to $\frac{dy}{dx} = \frac{y}{x}$.

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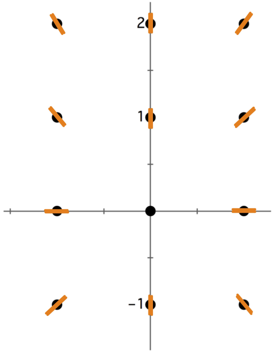
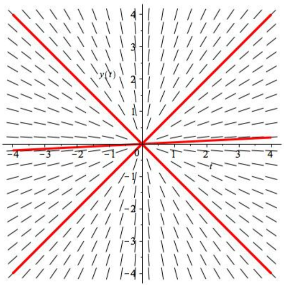




Clicker Question (11.1 and 11.4)

Assume separation of variables has given you

$$P(t) = \pm e^{-5t+c_1} =$$

Clicker Question (11.1 and 11.4)

Assume separation of variables has given you

$$P(t) = \pm e^{-5t+c_1} = \pm e^{-5t} e^{c_1} =$$

Clicker Question (11.1 and 11.4)

Assume separation of variables has given you

$$P(t) = \pm e^{-5t+c_1} = \pm e^{-5t} e^{c_1} = c_2 e^{-5t}$$

Solve for the solution when the initial condition is $P(0) = 1000$.

a) $P(t) = 1000e^{-5t}$

b) $P(t) = c_2 e^{-5000}$

c) $P(t) = ln(1000)e^{-5t}$

d) none of the above

Clicker Question (11.4)

Which of the following differential equations is NOT separable?

a) $\dfrac{dy}{dx} = \dfrac{3}{\ln y}$

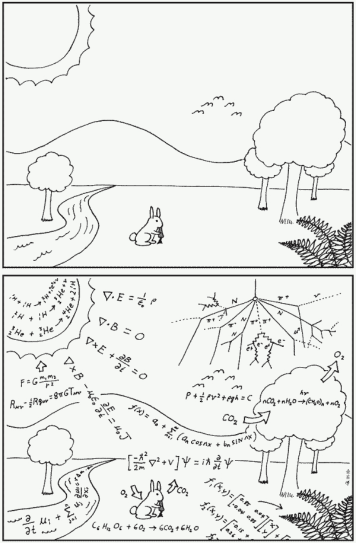
b) $\dfrac{dy}{dx} = 2x + y$

c) $\dfrac{dy}{dx} = e^{2x+y}$

d) $y' = 2x + 7$

e) $\sin 3x \; dx + 2y \cos^3 3x \; dy = 0$

Applications



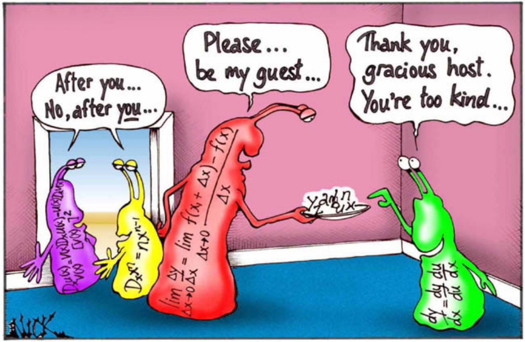
This is how scientists see the world.

https://abstrusegoose.com/strips/all_i_see_are_equations.PNG

Clicker Question (11.2 and 11.3)

Which will lead to a better graphical and numerical solution?

a) $\Delta x = .1$ and I have a good reason why
b) $\Delta x = .1$ but I am unsure of why
c) $\Delta x = .2$ but I am unsure of why
d) $\Delta x = .2$ and I have a good reason why
e) other

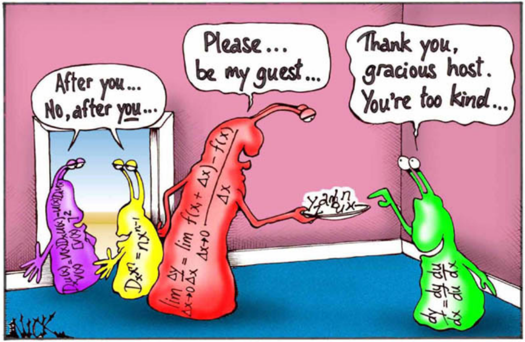


Deferential equations.

Clicker Question (11.2 and 11.4)

If $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, what does the slope field look like at (0,0)

a) horizontal
b) vertical
c) slope up to the right
d) slope down to the right



Deferential equations.

Clicker Question (11.1 and 11.4)

Many real-life objects grow and shrink proportional to the amount present. Is $y = sin(t)$ a solution to the DE $\frac{dy}{dt} = ky$?

a) yes and I have a good reason why
b) yes but I'm unsure of why
c) no but I'm unsure of why not
d) no and I have a good reason why not