11.5, 11.6 & 11.7: Applications of Differential Equations

Throw a ball in the air—where does it go? Put a pot on the stove—how does the heat spread? Blow a soap bubble—what shape will it take? Flip a switch and... a light turns on! All of these phenomena are governed by simple differential equations that have been known and studied for over 100 years. Nature at a small scale—electrons, protons and other particles—is governed by more complicated laws; these also involve differential equations. [Karen Uhlenbeck, Mathematics: The Equations of Nature]

Growth and Decay: Glacier

- Early stages, populations may grow or shrink rapidly at a rate proportional to the current population
  Example: In 2007 Grinnell Glacier in Glacier National Park covered about 142 acres and was estimated to be shrinking exponentially at a continuous rate of 4.3% of it's acreage each year.
- Define variables and write the differential equation
- When will the glacier be half of it's 2007 size?
- When will the glacier effectively disappear?

Decaying Temperature: Newton's Law of Cooling

You pour a cup of coffee at 180° and so Newton's law of cooling applies. Let T be the temperature and t be time.
Then Newton's law of cooling specifies that $\frac{dT}{dt} = -k(T - 72)$, where 72° is the temperature of the room.

proportional to difference in temperature
$T(t) = 72 + 108e^{-kt}$ was a solution as we saw in 11.1.
108 represented the difference between 180 (initial condition) and 72.

Clicker Question: Newton's Law of Cooling

Let T be the temperature of the cooling coffee and t be time. Newton's law of cooling specifies that $\frac{dT}{dt} = -k(T - 72)$, where 72° is the temperature of the room.

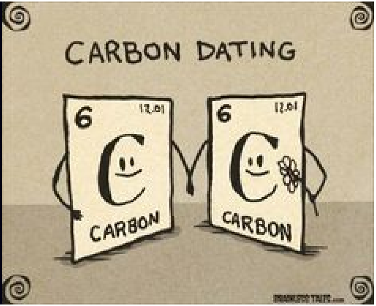
Solve the system by separation of variables method from 11.4.

What must we do here after we separate and integrate?

a) ln of both sides
b) e of both sides
c) algebraic manipulations of constants
d) more than one of the above but not all
e) all of a, b, c

Carbon-14 Dating

- Approximately what percentage of its carbon-14 should remain in a paining by Johannes Vermeer (1632–1675)? The half-life of carbon-14 is about 5730 years.

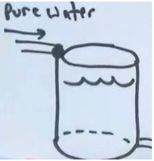



Girl with a Pearl Earring (1665)

Team led by Willard F. Libby developed radiocarbon method. Nobel Prize in Chemistry in 1960: *for his method to use Carbon-14 for age determinations in archaeology, geology, geophysics, and other branches of science*

Mixing Problems

A tank contains 100L of brine with a concentration of 0.2 kg of salt per liter. Pure water enters the tank at a rate of 10 L/min and the resulting solution, which is stirred continuously, runs out at the same rate.



- How many kilograms of salt remain after half an hour?
- Define variables and write the DE
- $\frac{dy}{dt}$ = rate of salt in - rate of salt out = 0 - rate of salt out

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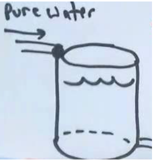

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- $\frac{dy}{dt}$ = rate of salt in - rate of salt out = 0 - rate of salt out
- When will the concentration reach .15 kg/L?

$$\frac{dy}{dt} = ky$$

Many objects grow and shrink proportional to the amount present

- Glacier
- Exponential growth at a continuous rate
- Compound interest: $\frac{dB}{dt} = rB$, where r is the interest rate, or other economic systems like those modeling labor
- Drug metabolism: $\frac{dA}{dt} = kA$, where k is negative
- Pollution in a lake: $\frac{dQ}{dt} = -r\frac{Q}{V}$, where r is the rate of flow, V is the volume of a lake, and no new pollution is introduced
- Snowball melts rate proportional to surface area (write r in terms of V)
- Sometimes all we know about a given system is its rate of change

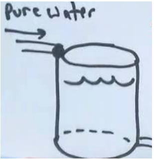
$$\frac{dT}{dt} = k(T - \text{surrounding temperature})$$

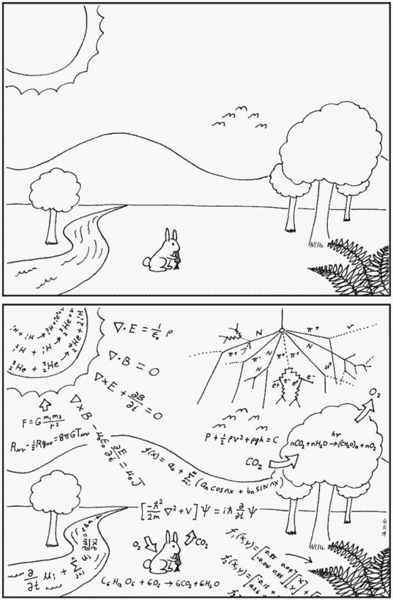
Heating and Cooling

- cooling: coffee, object in refrigerator, deceased body in forensic science...
- warming: cooking in oven...

$$\frac{dy}{dt} = \text{rate1} - \text{rate2}$$

- mixing problems: rate of salt in - rate of salt out
- population: birth rate - death rate
- weight : pounds/calorie (intake - maintenance)

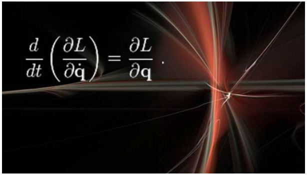




This is how scientists see the world.

DEs and PDEs: Existence and Uniqueness of Solutions?

Differential equations can be very hard to solve. In fact, a large number of them can't be solved at all! At least not numerically. Some qualitative information about the overall behavior of the system can sometimes still be learned—this is the field of study of dynamical systems [Karen Uhlenbeck, Mathematics: The Equations of Nature]



Euler-Lagrange equations: L measures energy in a physical system

Emmy Noether's theorem (roughly): Symmetric systems have corresponding conservation laws