Test	useful when	converges if	diverges if
Terms $\not\to 0$	$\sum a_n$, $a_n \not\to 0$ try this test first	n/a: $a_n \to 0$ inconclusive	$a_n \not\to 0$
Finite series		always	
Geometric	$\sum_0^\infty ax^n$ a starting value x constant ratio	$\|x\|<1$ to $\frac{a}{1-x}$	$\|x\| \geq 1$
Integral	pos, dec a_n known $\int$ ex: p-series $\sum \frac{1}{n^p}$	$\int^\infty a_n dn$ converges $\int$ bounds $\sum$	$\int^\infty a_n dn$ diverges
Linearity	$\sum a_n + b_n$	both conv	only 1 div

9.4: (Direct) Comparison Test

- For positive terms $a_n \leq b_n$
- If a series $\sum b_n$ converges, then so does any smaller (term by term) series
- If a series $\sum a_n$ diverges, then so does any larger (term by term) series
- Example: $\sum \frac{1}{ln(n!)}$

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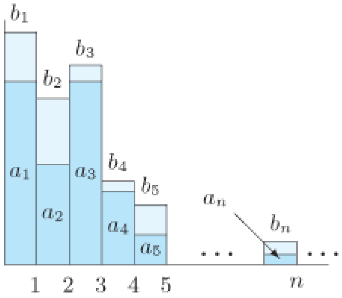
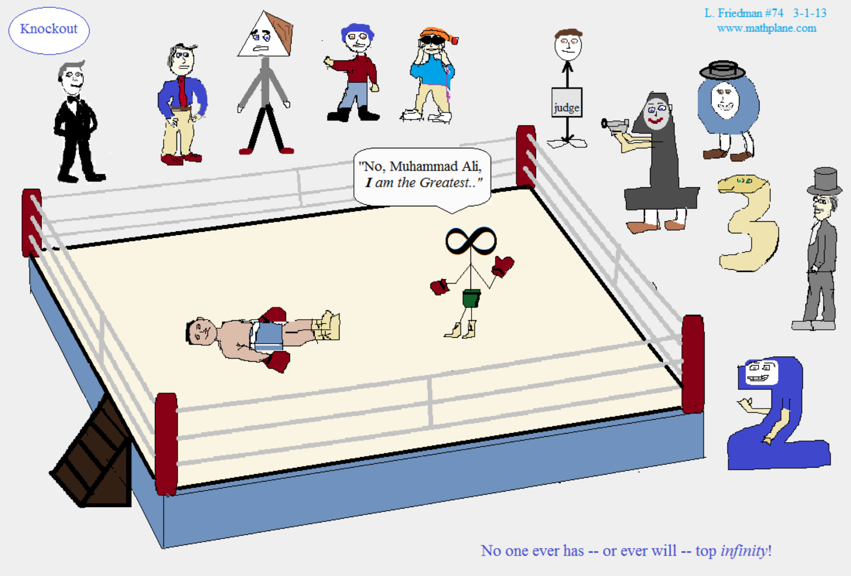
- Example: $\sum \frac{1}{ln(n!)}$ $\quad \frac{1}{ln(n!)} > \frac{1}{ln(n^n)} = \frac{1}{nln(n)}$




Knockout
L. Friedman #74 3-1-13
www.mathplane.com
judge
"No, Muhammad Ali, *I am the Greatest..*"
No one ever has -- or ever will -- top *infinity*!

Clicker Question

1. Can we use (direct) comparison on $\sum_{n=1}^{\infty} \frac{3}{n^5+n+8}$ by comparing it with $\sum_{n=1}^{\infty} \frac{3}{n^5}$? If so, does the sequence converge or diverge?

a) yes and they both converge
b) yes and they both diverge
c) no we can't use direct comparison on it but I don't know what else to use
d) no we can't use direct comparison on it but another method tells me it diverges
e) no we can't use direct comparison on it but another method tells me it converges

Clicker Question

2. Match the following series with the series we can use for the (direct) comparison test

Series 1: $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n(n^2+1)}$ Series 2: $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$

Comp 1: $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2}$ Comp 2: $\sum_{n=1}^{\infty} \frac{1}{n^3}$

a) Series 1 and Comparison 1, Series 2 and Comparison 2

b) Series 1 and Comparison 2, Series 2 and Comparison 1

c) We can use direct comparison on Series 1 with one of the comparison series but not Series 2

d) We can use direct comparison on Series 2 with one of the comparison series but not Series 1

e) We can't use direct comparison on either of these

9.4: Limit Comparison Test

- Can be difficult to identify $a_n \leq b_n$ for the direct comparison test. Sometimes easier to use limits!
- If $a_n > 0$, $b_n > 0$, and $0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum a_n$ behaves the same way as $\sum b_n$.

- Quotients of polynomials $\sum \frac{p(n)}{q(n)}$: try ratio of leading terms
- Example: $\sum \frac{n^2-4}{4n^3+3n-8}$

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WHERE IS THE LIMIT?®

- Quotients of polynomials $\sum \frac{p(n)}{q(n)}$: try ratio of leading terms
- Example: $\sum \frac{n^2-4}{4n^3+3n-8}$

  Compare to $\sum \frac{n^2}{4n^3} = \sum \frac{1}{4n}$, which diverges by integral test.

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} =$

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  $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \ldots = \lim\limits_{n\to\infty} \frac{1-\frac{4}{n^2}}{1+\frac{3}{4n^2}-\frac{8}{4n^3}} =$

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$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ldots = \lim_{n\to\infty} \frac{1-\frac{4}{n^2}}{1+\frac{3}{4n^2}-\frac{8}{4n^3}} = 1 \qquad 0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

Clicker Question

3. Use the limit comparison test on $\sum_{n=1}^{\infty} \frac{4n^2 + n + 9}{7n^3 + 13}$. Does it converge?

a) yes and I have a good reason why
b) yes but I am unsure of why
c) no, but I am unsure of why
d) no, and I have a good reason why
e) I was unable to use the limit comparison test

9.4: Ratio Test

- For $\sum a_n$, if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$
  $L < 1$ implies convergence
  $L > 1$ implies divergence
  $L = 1$ gives no information.
- Often useful when we have factorials (n!) or exponents in the series because the algebra will let us cancel
- Even though the ratio between terms is not constant like in geometric series, if it is manageable ($L < 1$) then the series converges

Test	useful	converges if	diverges if
Terms $\not\to 0$	$\sum a_n$, $a_n \not\to 0$ try this test first	n/a: $a_n \to 0$ inconclusive	$a_n \not\to 0$
Finite series		always	
Geometric	$\sum_0^\infty ax^n$, const x	$\lvert x\rvert<1$ to $\frac{a}{1-x}$	$\lvert x\rvert \geq 1$
Integral	pos, dec a_n known $\int$ ex: p-series $\sum \frac{1}{n^p}$	$\int^\infty a_n dn$ converges $\int$ bounds $\sum$	$\int^\infty a_n dn$ diverges
Linearity	$\sum a_n + b_n$	both conv	only 1 div
Comparison	conv by larger $\sum$ div by smaller $\sum$	$0 \leq a_n \leq b_n$ $\sum b_n$ conv	$0 \leq b_n \leq a_n$ $\sum b_n$ div
Limit Comp	$0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$ pos terms	$\sum b_n$ conv same behavior	$\sum b_n$ div
Ratio	$\lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} \neq 1$	$L < 1$	$L > 1$

Clicker Question

4. Which test will help you determine if the series $\sum e^{-n}$ converges or diverges?

a) geometric series
b) integral test
c) ratio test
d) more than one of the above but not all
e) all of a, b and c

9.4: Absolute Value Test for Alternating Series

- *Absolute Value Test*: If $\sum |a_n|$ converges, then so does $\sum a_n$.

$$\sum_{n=1}^{\infty} \frac{1}{2}^n \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

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$$\sum_{n=1}^{\infty} \frac{1}{2}^n \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$$

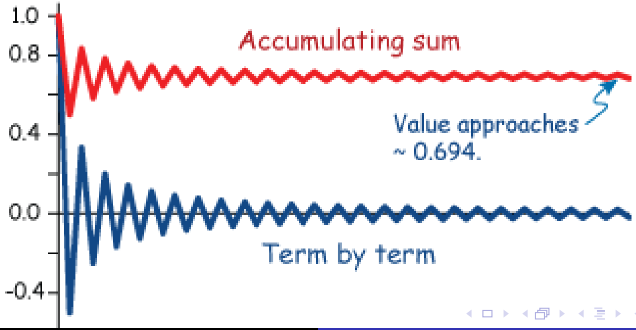
Caution: reverse implication is false!

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

9.4: Alternating Series Test

- Decreasing terms, alternating terms
- If $|a_n| \geq |a_{n+1}|$ and $\lim_{n\to\infty} |a_n| = 0$, then the alternating series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$



Clicker Question

4. Which are valid arguments?

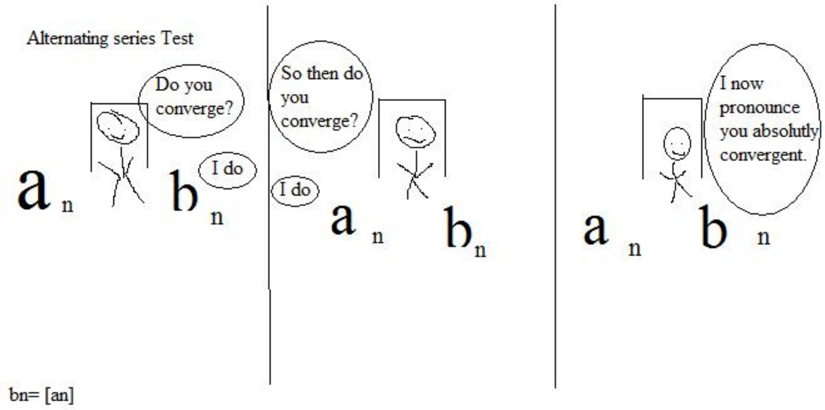
a) $\sum \frac{4}{n}$ converges because $\lim_{n\to\infty} \frac{4}{n} = 0$

b) $\sum (-1)^n n^2$ converges by the alternating series test

c) both are valid

d) neither are valid

9.4: Absolute or Conditionally Convergent?



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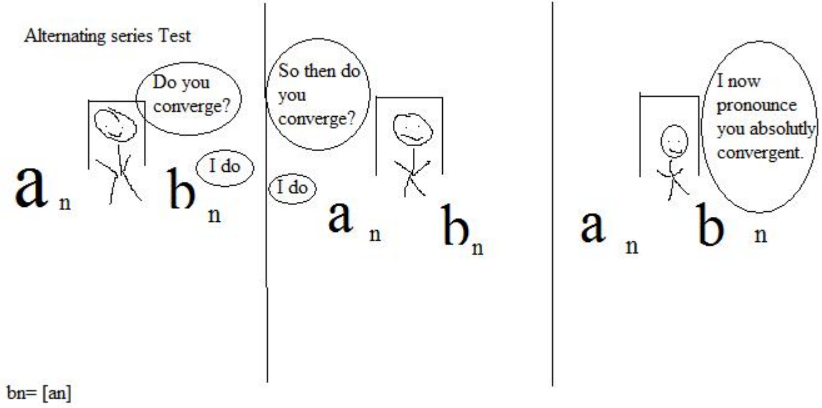
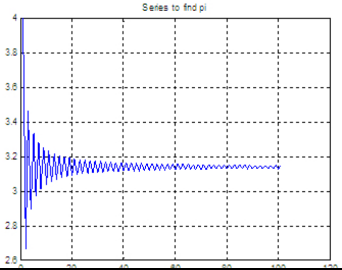


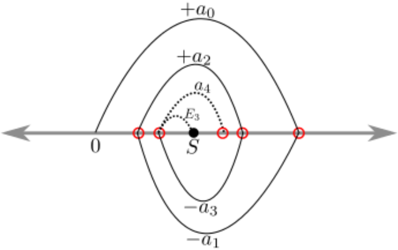
Conditionally Convergent: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Alternating Series Estimation

- Truncation error of using S_n, the partial sum, to approximate an alternate series, is less than the absolute value of next term in the series a_{n+1}
- Example $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$

  S_9 has an error bounded by $\frac{1}{(9+1)^2}$





History

- ratio test: first published by Jean le Rond d'Alembert and is sometimes known as d'Alembert's ratio test or as the Cauchy ratio test
- alternating series test was used by Gottfried Leibniz and is sometimes known as Leibniz's test, Leibniz's rule, or the Leibniz criterion

- widely used in mathematics and other quantitative disciplines such as physics, computer science, & finance.

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	try this test first	n/a: $a_n \to 0$	
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	known $\int$	converges	diverges
	ex: p-series $\sum \frac{1}{n^p}$	$\int$ bounds $\sum$	
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Comparison	conv by larger $\sum$	$0 \leq a_n \leq b_n$	$0 \leq b_n \leq a_n$
	div by smaller $\sum$	$\sum b_n$ conv	$\sum b_n$ div
Limit Comp	$0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$	$\sum b_n$ conv	$\sum b_n$ div
	pos terms	same behavior	
Ratio	$\lim_{n\to\infty} \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert} \neq 1$	$L < 1$	$L > 1$
Alternating	alternating terms	$\lim_{n\to\infty} \lvert a_n\rvert = 0$	$\lim_{n\to\infty} \lvert a_n\rvert \neq 0$
	$\lvert a_n\rvert$ decreasing		
Absolute	alternating terms	$\sum \lvert a_n\rvert$ conv	inconclusive