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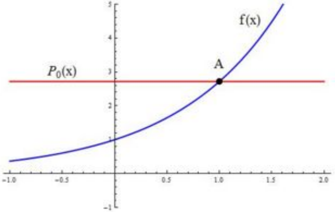

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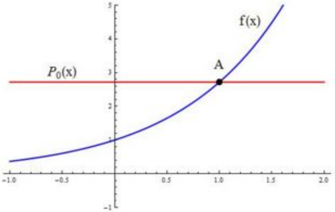


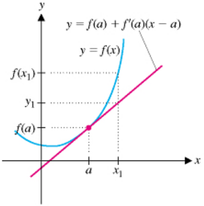

- Degree 1 Taylor polynomial: $f(a) + f'(a)(x - a)$ [slope]

Picture credit: http://dev.quillandpad.com/2014/02/10/russia-spaceflight-and-watches-paving-the-way-for-ski-jumping-since-1930/

10.1 Approximating $f(x)$: Degree n Taylor Polynomial

- Polynomials are easy to work with, so approximate $f(x)$
- To get a better match, use more derivatives!
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- Coefficient of $(x-a)^3$? $\frac{f^{(3)}(a)}{3!}$
- $f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$
  $= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

Picture credit:

http://zazzleabies.blogspot.com/2013/08/nerd-baby-stuff-brought-to-you-by.html

10.1 Approximating $f(x)$: Degree n Taylor Polynomial

Degree n polynomial includes all the terms up to and including the one with x^n. Some of the terms might be zero.

$$\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

n $\quad$ $f^{(n)}(x)$ $\quad$ $f^{(n)}(a)$ $\quad$ Taylor Term $\frac{f^{(n)}(a)}{n!}(x-a)^n$

Picture credit:

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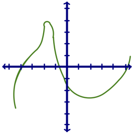
Clicker Question

1. What is the sign of $(x-2)^2$ (concavity) in the Taylor polynomial of degree 4 for $f(x)$ about $x = 2$, given the graph below (assume the coefficient of that term is not zero).



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b) negative

c) no way to tell without more information about $f(x)$

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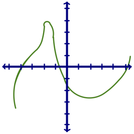

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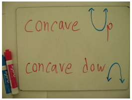
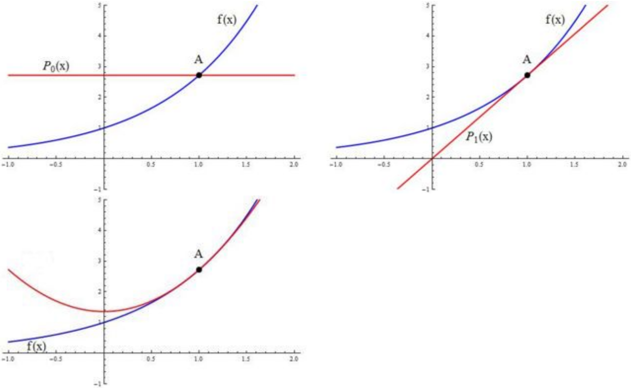


Picture credit: `https://mathwithbaddrawings.com/2013/05/30/how-to-remember-concavity/`

By the signs of the corresponding terms, the first three terms (if they are nonzero) tell us

- whether the function is above or below the $x-$axis at a
- whether the function is increasing or decreasing at a
- whether the function is concave up or down at a:

Degree 0, 1, 2 Taylor polynomials:



Clicker Question

2. Without help from technology and without any by-hand calculation, is there an instantaneous way to know the 4th degree Taylor polynomial of the polynomial $f(x) = -x + x^2 + x^4$ at $x = 0$?

a) yes and I know what it is
b) yes but I'm unsure what it is
c) no shortcuts here that I see

Taylor 'humor':
Q: Why would Taylor never run for US president?

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Taylor 'humor':
Q: Why would Taylor never run for US president?
A: He doesn't want to be limited to two terms.
[Sarah Zureick-Brown]

10.2: Taylor Series for $f(x)$ at $x = a$

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i \text{ or } \sum_{i=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n =$$

$$f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$$

I have computed this Table so far, that the Reader may see in what manner this Method Approximates; this whole Work, as it appears, costing a little more than three Hours time. [Brook Taylor(1685–1731) Approximating Roots of Equations, 1717]

- Taylor Series is a power series, so we use the ratio test or geometric series to investigate convergence and r
- $a = 0$ is called the Maclaurin series, named for Scottish mathematician Colin Maclaurin (1698–1746)

Picture credit: `https://www.npg.org.uk/collections/search/portrait/mw06208/Brook-Taylor`

Taylor Series Convergence?

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- Taylor Series is a power series, so we use the ratio test or geometric series to investigate convergence and R
- If they converge do they converge to $f(x)$? Maybe.
- Example 1: Taylor series for $\frac{1}{1-x}$ at $x = 0$?
- Example 2: Taylor series for $\sin x$ at $x = 0$?

In the best of worlds all functions should be polynomials. Since this fails, one uses Taylor's formula to replace (some) functions by polynomials which reasonably approximate them. Mathematicians who can't face these harsh realities become algebraic geometers and live in a dream world where indeed all functions are polynomials. [Georges Elencwajg]

Picture credit: `http://www.nerdytshirt.com/series-premiere.html`

Commonly Used Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ...$$

- $r = \infty$ (ie for all x):
  - $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ even
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ odd
- $r = 1$
  - $\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$ geometric
  - $(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \binom{p}{n} x^n$

    binomial (we won't focus on this one)

Group Work Target Practice

Identify this as a Taylor series of a known function and substitute for x to find the sum of $1 - \frac{3^2}{2!} + \frac{3^4}{4!} + \ldots$.
1) Which is it?
2) What do the other Taylor series look like?

a) e^3

b) $\cos(3)$

c) $\sin(3)$

d) $\frac{1}{1-3}$

Taylor's PR slogan?

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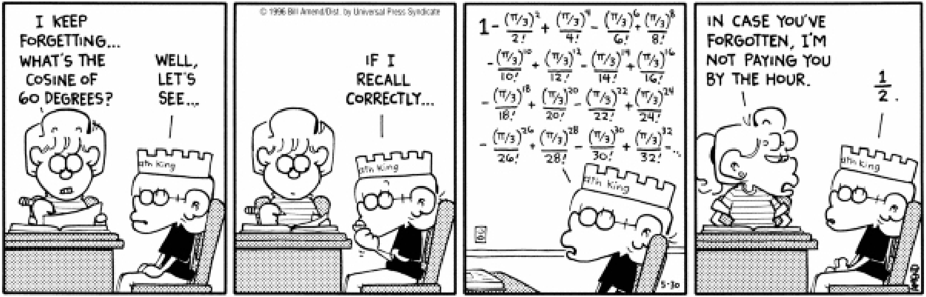
Taylor's PR slogan?

infinite sums approximated in finite time*

*some restrictions apply [Sarah Zureick-Brown]



FoxTrot by Bill Amend, University Press Syndicate

Group Work Target Practice

Compute the Taylor series for $f(x) = \ln x$ at the point $a = 1$.

$\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ or equivalently $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

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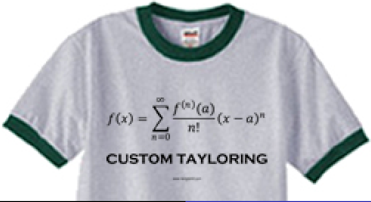
$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$$

Picture credit: `http://www.nerdytshirt.com/custom-tayloring.html`



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