Chapter 11 DEs
Group Work Target Practice

1. Hydrocodone bitartrate is used as a cough suppressant. After the drug is fully absorbed, the quantity in the body decreases at a rate proportional to the amount left in the body. The half-life of hydrocodone bitartrate in the body is 3.8 hours, and the usual dose is 10 mg.

a. Write a DE for the quantity, $Q(t)$, of hydrocodone bitartrate in the body at time t, in hours, since the drug was absorbed.

$\frac{dQ}{dt} = -kQ$, where the constant of proportionality, k is positive [or equivalently.

b. Find the equilibrium solution of the DE—the constant of proportionality is assumed to be nonzero. Based on the context, do you expect the equilibrium to be stable or unstable?

An equilibrium solution is when $\frac{dQ}{dt} = 0$ and stays that way. Then set $\frac{dQ}{dt} = -kQ = 0$, but k is nonzero, so $Q = 0$. This means there is no drugs in the body. We expect this to be a stable solution, since Hydrocodone bitartrate leaves the body eventually.

c. Write the initial condition using the usual dose.

The usual dose is 10 mg at time 0, so $Q(0) = 10\text{mg}$

d. Write a half-life condition using the half-life (the time taken to fall to half its usual dose).

$Q(3.8) = 5\text{mg}$, half of the starting dose

e. Here is how we would solve the DE and use the usual dose as the initial condition.

Separate: $\frac{dQ}{Q} = -kdt$

Integrate: $\ln|Q| = -kt + c$

Solve for Q, which is positive, so we can drop the absolute value sign, and exponentiate both sides: $Q = e^{-kt+c} = e^{-kt}e^{c} = c_2 e^{-kt}$

The usual dose is 10 mg at time 0, so plug in to solve for c_2: $10 = Q(0) = c_2 e^0 = c_2$. Then $Q = 10e^{-kt}$

f. Here is how we would use the half-life to find the constant of proportionality.

$Q(3.8) = 5\text{mg}$, half of the starting dose:

$5 = 10e^{-k3.8}$ so $\frac{1}{2} = e^{-k3.8}$. Take ln of both sides: $\ln\frac{1}{2} = \ln e^{-k3.8}$.

$\ln 1 - \ln 2 = -k3.8$

$-\ln 2 = -k3.8$

$\frac{\ln 2}{3.8} = k$, so $k \approx .182$

g. We can then answer questions like, how much of the 10mg dose is still in the body after 12 hours?

$Q(12) = 10e^{-.182 \cdot 12} \approx 1.126\text{mg}$

2. Write the differential equations and any initial and additional conditions:

a. A 20° (Celsius) yam is put in a 200° oven. Assume that the temperature of the yam is 120° after 30 minutes. What will the temperature be after 50 minutes?

Let $Y(t)$ be the yam temperature in celsius at time t in minutes.

The DE is $\frac{dY}{dt} = -k(Y - 200)$ where $k > 0$ (note that $Y < 200$, so Y-200 is negative and hence when multiplied by -k will give a positive derivative, which makes sense since the yam is heating up, so increasing temperature). The initial condition is $Y(0) = 20$, and we are also given $Y(30) = 120$, and asked to find $Y(50)$.

Separate: $\frac{dY}{Y-200} = -kdt$

Integrate: $ln|Y - 200| = -kt + c$

Solve for Y: $200 - Y = |Y - 200| = e^{-kt+c} = c_2 e^{-kt}$ (notice that Y is less than 200, so the absolute value is the other way).

$Y = 200 - c_2 e^{-kt}$.

Plug in the initial condition to solve for c_2: $20 = Y(0) = 200 - c_2 e^0 = 200 - c_2$, so $c_2 = 180$ and $Y = 200 - 180e^{-kt}$.

Use the condition that $Y(30) = 120$ to solve for k:

$120 = Y(3) = 200 - 180e^{-k \cdot 30}$

$-80 = -180e^{-k \cdot 30}$

$\frac{4}{9} = e^{-k \cdot 30}$

$\ln \frac{4}{9} = \ln e^{-k \cdot 30} = -k \cdot 30$

$k = -\ln \frac{4}{9}/30 \approx .027$.

Plug in 50 to solve for the temperature: $Y(50) = 200 - 180e^{-.027 \cdot 50} \approx 153.3°$

b. A detective finds a deceased individual at 9am. The temperature of the body is measured at 90.3° (Fahrenheit). One hour later, the temperature is 89°. Assume the temperature of the room has been maintained at a constant 68°. Estimate the time of death.

Let $T(t)$ be the temperature in Fahrenheit at time t in hours.

The DE is $\frac{dT}{dt} = -k(T - 68)$ where $k > 0$. The initial condition is $T(0) = 90.3$, and we are also given $T(1) = 89$, and asked to find t when $T(t) = 98.6$.

Separation of variables and integration as in part a gives

$T = 68 + c_2 e^{-kt}$.

Plug in the initial condition $T(0) = 90.3$ to solve for $c_2 = 22.3$.

Use the condition that $T(1) = 89$ to solve for k: $89 = 68 + 22.3e^{-k \cdot 1} ... k = -\ln(21/22.3) \approx .060064$

So $T(t) \approx 68 + 22.3e^{-.060064t}$.

Set $T(t) = 98.6$ and solve for t:

$68 + 22.3e^{-.060064t} \approx 98.6$

$e^{-.060064t} \approx 30.6/22.3$ Take ln both sides: $t \approx \ln(30.6/22.3)/(-.060064) \approx -5.27$ hours (ie before 9am), so 3:45 am, approximately.

c. At 1pm there is a power failure, which is bad news for your electric heater. Assume it was 68° (Fahrenheit) when the power went out in the house, and it is 10° outside. At 10PM it is 57°. If the outdoor temperature remains constant, what temperature will it be at 7am the next morning? Should you worry about your water pipes freezing?

Let $T(t)$ be the temperature in Fahrenheit at time t in hours.

The DE is $\frac{dT}{dt} = -k(T - 10)$ where $k > 0$. The initial condition is $T(0) = 68$, and we are also given $T(9) = 57$, and asked to find $T(18)$.

Separation of variables and integration as in part a gives

$T = 10 + c_2e^{-kt}$.

Plug in the initial condition $T(0) = 68$ to solve for $c_2 = 58$.

Use the condition that $T(9) = 57$ to solve for k. See part b, which is similar for this portion.

$k = -1/9ln(47/58) \approx .0234$

So $T(t) \approx 10 + 58e^{-.0234t}$.

At 7am, after 18 hours from 1pm, $T(18) \approx 10 + 58e^{-.0234 \cdot 18} \approx 48°$, so the pipes won't freeze.

3. Write a differential equation for the balance in an investment fund with time measured in years when the balance is losing value at a continuous rate of 6.5% per year, and payments are being made out of the fund at a continuous rate of $50,000 per year.

Let $P(t)$ be the principal in dollars at time t in years.

The DE is $\frac{dP}{dt} = -.065P - 50000$

4. Write a differential equation for $\frac{dS}{dt}$ in kg/min, where S is the salt in kg and t is in min: A tank containing salt mixed into water has salt added to the tank at the rate of 0.1 kg/min. The contents of the tank are kept thoroughly mixed, and the contents flow in and out at 10 liters/min. The tank contains 100 liters of water.

$\frac{dS}{dt}$ = rate of salt in - rate of salt out

Salt goes in at 0.1 kg/min.

Salt goes out at S kg / 100 liters $\times$ 10 liters/min of contents flowing out. Notice the tank has 100 liters at all times because the output liters/min is the same as the input.

$\frac{dS}{dt} = .1 - .1S$