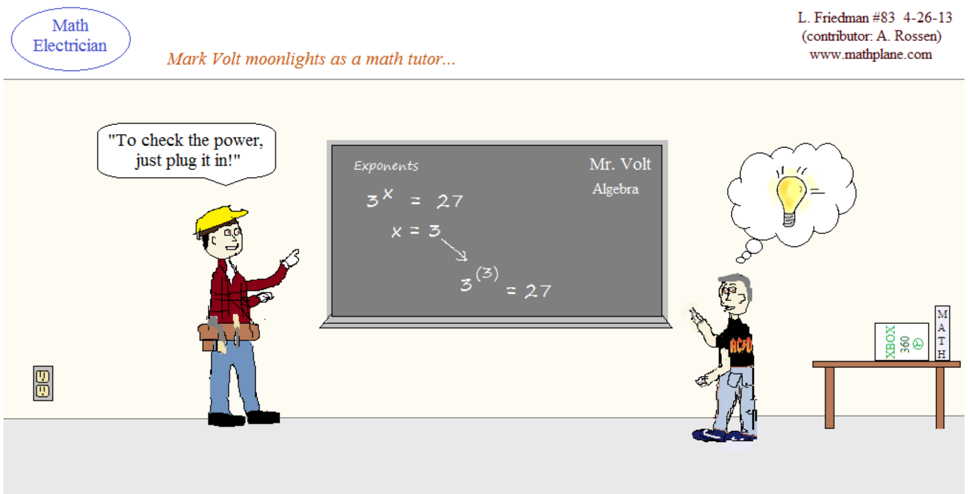
Chap 11

1. You pour a cup of coffee at 180° and so Newton's law of cooling applies. Let T be the temperature and t be time. Then Newton's law of cooling specifies that the differential equation is $\dfrac{dT}{dt} = -k(T - 72)$, where 72° is the temperature of the room. Is $T(t) = 72 + 108e^{-kt}$ a solution to the differential equation?

   11.1 really does come down to this math comic:



Alex gets the connection!

   We check by plugging in to $\dfrac{dT}{dt} = -k(T - 72)$—we'll need the derivative of $T(t) = 72 + 108e^{-kt}$ for the left side, so use chain rule: $T'(t) = 0 + 108e^{-kt}(-k)$. Now we plug in the derivative to the left side and the proposed solution to the right side, and we see if they satisfy the differential equation:

   $108e^{-kt}(-k) = \dfrac{dT}{dt} \stackrel{?}{=} -k(T - 72) = -k(72 + 108e^{-kt} - 72) = -k108e^{-kt}$.

   These are equal, so it is a solution.

   Notice that 108 actually comes from the difference between the coffee and room temperature: 180-72 = 108. These form the basis of many real-life growth and decay problems, including Newton's law of cooling and population modeling, to name a few.

2. Which of the following is a solution to $y' = ky$? $y(t) = e^{kt}$, $y(t) = e^{kt+5} = e^{kt}e^5$, $y(t) = 2e^{kt}$?

   We check by plugging each proposed solution into $y' = ky$ to see if it works–we'll need the derivative for the left side, so we'll use chain rule:

   a) $y(t) = e^{kt}$ $\quad ke^{kt} = y' \stackrel{?}{=} ky = ke^{kt}$. Yes it is a solution.

   b) $y(t) = e^{kt+5} = e^{kt}e^5$ $\quad ke^{kt+5} = y' \stackrel{?}{=} ky = ke^{kt+5}$. Yes it is a solution.

   c) $y(t) = 2e^{kt}$ $\quad k2e^{kt} = y' \stackrel{?}{=} ky = k2e^{kt}$. Yes it is a solution.

3. Do either of $y(t) = e^{kt+2}$ or $y(t) = 2e^{kt}$ work when $y(0) = 2$?

   We saw above that each of these are solutions, but the question here is whether they solve the given initial condition. Since $y(0) = 2$, we plug in t=0 and see whether y=2:

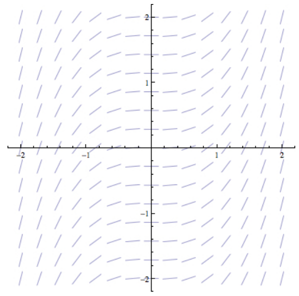
   a) $y(t) = e^{kt+2}$ $\quad y(0) = e^{k0+2} = e^2 \neq 2$, so this is not a solution for $y(0) = 2$.

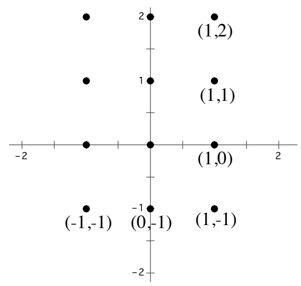
   b) $y(t) = 2e^{kt}$ $\quad y(0) = 2e^{k0} = 2e^0 = 2 \cdot 1 = 2$, so yes this is a solution for this initial condition.

4. If $\dfrac{dy}{dx}$ is 0 at some point, what does that tell you about the tangent line at that point?

   The slope of the tangent line is 0, which means the change in y is 0, so it is horizontal.

5. Which differential equation(s) correspond to the slope field?





   Which differential equation(s) correspond to the slope field, $\frac{dy}{dx} = xy$ or $\frac{dy}{dx} = x^2$? The second one as the slope field is either horizontal (0 slope) or sloping up to the right (positive slope). So that can only be $\frac{dy}{dx} = x^2$, of the two. Conversly, if you plot some slope field points directly, you can see that $\frac{dy}{dx} = xy$ has negative slopes in quadrant 2 and 4.

6. Apply Euler's method one time on $\frac{dy}{dx} = (x-2)(y-3)$ with $\Delta x = .1$, starting at the point $(0, 4)$.

   The idea of Euler's method is:

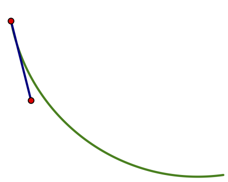
   - Calculate the slope of the initial point via the DE
   - Head off a small distance Δx (fixed) in that direction to $(x_0 + \Delta x, y_0+\text{slope } \Delta x)$
   - (when directed) Use the new signpost—recalculate the slope from the DE, using the new point...

   Starting at $(0, 4)$

x	y	$\frac{dy}{dx}$	$\Delta y = \text{slope}\Delta x$	$(x + \Delta x, y + \Delta y)$	new y
0	4	$(0-2)(4-3) = -2$	$-2 \cdot .1 =$ -.2	$(0 + .1, 4 - .2)$	3.8

7. If a function is decreasing and concave up at (x_0, y_0), what, if anything, can we say about $(x_0 + \Delta x, y_0+\text{slope } \Delta x)$?

   Sketch the graph and the linear approximation to see that it will underestimate the true value of the function, because as long as the function was not linear it will be below it.



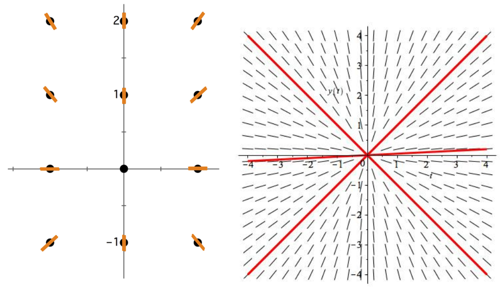
8. If we separate the variables in the differential equation $3x\frac{dy}{dx} = y^2$ we can obtain:

   Place all the y variables on the left side using division of y^2 and all the x variables on the right side using multiplication of dx and division of x. Separation of variables can look like $3y^{-2}dy = \dfrac{dx}{x}$, where I wrote it to get ready for the integration step (power rule on the left side and logarithm on the right).

9. Use separation of variables to find the solution to $\dfrac{dy}{dx} = \dfrac{y}{x}$.

$\dfrac{dy}{dx} = \dfrac{y}{x}$ becomes $\dfrac{dy}{y} = \dfrac{dx}{x}$. Then we integrate both sides to obtain $\ln|y| = .\ln|x| + c$. To solve for y, we exponentiate both sides: $e^{\ln|y|} = e^{\ln|x|+c_1}$. Next we can use some algebraic simplifications: $|y| = e^{\ln|x|+c_1} = e^{\ln|x|}e^{c_1} = |x|e^{c_1} = |x|c_2$

It is not surprising that this is solution is linear, where the object's y-value is a constant times the x-value, as we previously solved this using slope fields, by plugging in for values on the grid:



10. Assume separation of variables has given you $P(t) = \pm e^{-5t+c_1} = \pm e^{-5t}e^{c_1} = c_2 e^{-5t}$

Solve for the solution when the initial condition is $P(0) = 1000$.

We see that $t = 0$ and $P = 1000$ so we plug those into $P(t) = c_2 e^{-5t}$:

$1000 = P(0) = c_2 e^{-5\cdot 0} = c_2 \cdot 1$, since $e^0 = 1$, so $c_2 = 1000$ and the general solution is $P(t) = 1000e^{-5t}$

11. Which of the following differential equations is NOT separable?

a) $\dfrac{dy}{dx} = \dfrac{3}{\ln y}$

b) $\dfrac{dy}{dx} = 2x + y$

c) $\dfrac{dy}{dx} = e^{2x+y}$

d) $y' = 2x + 7$

e) $\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$

We can separate all of these except $\frac{dy}{dx} = 2x + y$ as there is no way to move all the x pieces on one side and separate them from the y because the left side has a quotient but the right side has a sum.

Part a we can use multiplication by $\ln y$ and dx to separate the variables (then the left side would be special parts for integration), in Part c we can use exponent rules to separate the exponentials and then use multiplication and division in order to obtain $\frac{dy}{dx} = e^{2x+y} = e^{2x}e^{y}$, so $dye^{-y} = e^{2x}dx$ (and then integrate by w-subs). In Part c the xs are already on the right side, so move the dx in $y' = \frac{dy}{dx}$ over by multiplication, and in the last one, Part e, use a combination of subtraction, multiplication and division to separate the variables (another w-subs would follow).

12. What will lead to a better numerical solution, $\Delta x = .1$ or $\Delta x = .2$?

The smaller the change in x, the better the numerical solution.

As we talked about with Taylor series, if we are very very close to the center, the starting point here, a Taylor polynomial like the linear approximation looks like the function. It is the Lagrange Error estimate/Taylor's Theorem that shows as that as we are closer to the center, the error is smaller. In this case, where the linear approximation has $n = 1$, we have:

$$\frac{M}{(1+1)!}(x-a)^{1+1}$$

When $\Delta x = .1$ versus $\Delta x = .2$, then $(x - a) = \Delta x$, so we see that the smaller Δx gives the smaller error. Because of the power of 2, the local error is proportional to the square of the step size.

13. If $\frac{dy}{dx} = \frac{1}{1+x^2}$, what does the slope field look like at (0,0)

    We plug into the right side of the differential equation, in this case using the x value since the DE only has x values on the right: $\frac{1}{1+0^2}$. So the slope is 1 and the slope field goes up to the right. If you separate this equation and integrate you'll see that $y = arctan(x)$, which indeed does slope up to the right at the origin.

14. Many real-life objects grow and shrink proportional to the amount present. Is $y = \sin(t)$ a solution to the DE

    $$\frac{dy}{dt} = ky?$$

    No—the solution would be exponential. Here we can test by plugging in to see $y = sin(t)$ won't work:

    $\frac{dy}{dt} = ky = \frac{d(\sin(t))}{dt} = ksin(t)$, but then the derivative gives $\cos(t) = k\sin(t)$ which is impossible no matter the k value.