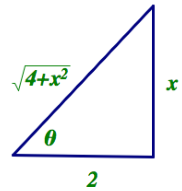
Solutions to Trig Substitution Group Target Practice:

1. Use a trig substitution to transform the integral $\int \sqrt{4+x^2}\,dx$. Completely reduce the integrand, but **do not integrate**.

$x = 2\tan\theta$ as in the following diagram: 

. x is always the opposite side in the trig substitution triangles. When the radical has plus in it, it is on the hypotenuse, because that meshes with the Pythagorean (when it has minus in it, it will be the adjacent side).

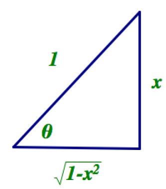
Then $dx = 2\sec^2\theta\,\mathrm{d}\theta$

Sub both in: $\int \sqrt{4+x^2}\,dx = \int \sqrt{4+(2\tan\theta)^2}\,2\sec^2\theta\,\mathrm{d}\theta$

Reduce: $= \int \sqrt{4+4\tan^2\theta}\,2\sec^2\theta\,\mathrm{d}\theta = \int \sqrt{4(1+\tan^2\theta)}\,2\sec^2\theta\,\mathrm{d}\theta$

$= \int \sqrt{4(\sec^2\theta)}\,2\sec^2\theta\,\mathrm{d}\theta = \int 2\sec\theta\,2\sec^2\theta\,\mathrm{d}\theta = \int 4\sec^3\theta\,\mathrm{d}\theta$

2. Compute $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

(a) & (b) $x = \sin\theta$ as in the following diagram: 

. x is always the opposite side in the trig substitution triangles. When the radical has minus in it, it is the adjacent side because that meshes with the Pythagorean (when it has plus in it, it will be the hypotenuse).

Then $dx = \cos\theta\,\mathrm{d}\theta$

(c) Sub both in to convert the integral to one with θ and simplify:

$$\int \frac{(\sin\theta)^2}{\sqrt{1-(\sin\theta)^2}}\cos\theta\,\mathrm{d}\theta = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta = \ldots = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,\mathrm{d}\theta = \int \sin^2\theta\,\mathrm{d}\theta$$

(d and e) Use the half angle formula so that we can integrate:

$\int \sin^2\theta\,\mathrm{d}\theta = \int \frac{1}{2} - \frac{1}{2}\cos(2\theta)\,\mathrm{d}\theta = \int \frac{1}{2}\,\mathrm{d}\theta - \int \frac{1}{2}\cos(2\theta)\,\mathrm{d}\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + c$, where $w = 2\theta$ is used on the second integral for $w-$subs

(f) Use the double angle formula to eliminate a double angle

$=\frac{1}{2}\theta - \frac{1}{4}2\sin(\theta)\cos(\theta) + c = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta) + c$

(g) What conversion formulas will you need to get back to x?

$\sin(\theta) = \frac{x}{1} = x, \cos(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}, \theta = \arcsin x$

(h) Apply the conversion formulas:

$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta) + c = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + c$