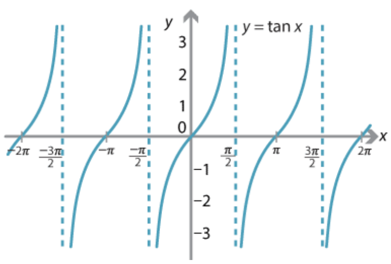
7.6 Improper Integrals
Group Work Target Practice

1. Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos(x)^2}\, dx$.

   Oops—$\cos(x)$ is 0 at $\frac{\pi}{2}$, making the denominator 0, so the integral is improper. Convert using the definition of $\sec(x)$, because this turns this integral into a known one from calc 1. So we write out the integral on a finite region using limits:

   $\lim_{b \to \frac{\pi}{2}^-} \int_0^b \sec^2(x)\, dx$. Then we integrate: $\lim_{b \to \frac{\pi}{2}^-} \tan(x)\Big|_0^b$. Then we plug in the endpoints: $\lim_{b \to \frac{\pi}{2}^-} \tan(b) - \tan(0)$.

   This improper integral diverges because $\tan(x)$ does not tend to a finite number as $b \to \frac{\pi}{2}^-$ as per the graph below:



2. Evaluate $\int_0^{\infty} xe^{-x}\, dx$. Hint: You'll need to apply L'Hôpital's rule at the end.

   This integral is improper because it has ∞ in the integral limits, so we write it as a limit: $\lim_{b \to \infty} \int_0^b xe^{-x}\, dx$. This is integration by parts. It is not $w-subs$ because we don't have a function inside another function and its derivative (up to a scalar) to make use of (because we are missing an x to pair with dx for $w = -x^2$ and $dw = -2xdx$). We can use parts because we have a product of two different functions that fit into detail: exponential e^{-x} is before algebraic x, so the derivative (d) from detail shows $v' = e^{-x}$.

   $u = x \qquad v' = e^{-x}$

   $u' = 1 \qquad v = -e^{-x}$ (if you don't understand the -, write out $w = -x$ for $w - subs$).

   $$uv - \int u'v dx = \lim_{b \to \infty} -xe^{-x}\Big|_0^b - \int -e^{-x} dx = \lim_{b \to \infty} -xe^{-x}\Big|_0^b - e^{-x}\Big|_0^b$$

   Plug in b and subtract 0 plugged in:

   $= \lim_{b \to \infty} -be^{-b} - -0e^{-0} - e^{-b} - -e^{-0} = \lim_{b \to \infty} -\frac{b}{e^b} + \frac{0}{1} - \frac{1}{e^b} + 1$

   We'll need L'Hôpital's on $-\frac{b}{e^b}$ which tends to $\frac{\infty}{\infty}$, so take the derivative of the numerator and (separately) the denominator:

   $= \lim_{b \to \infty} -\frac{1}{e^b} + \frac{0}{1} - \frac{1}{e^b} + 1.$

   exponentials get large, so one over them get smaller and smaller, and hence the integral converges to 0 + 0 + 0 +1 =1.