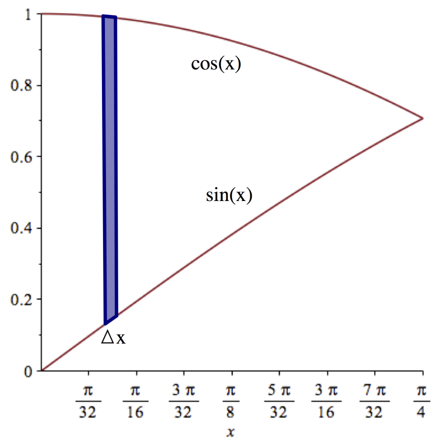
8.1 and 8.2 Area and Volume by Slicing Group Solutions

1. (a) Sketch a graph of the functions to find the enclosed region between $\sin(x)$ and $\cos(x)$ from 0 to $\frac{\pi}{4}$ and the Riemann slice.



   (b) Base of the rectangle? Δx

   (c) Which function is larger in that variable (top for x, right for y)? $\cos(x)$

   (d) What is the height of the rectangle (top-bottom or right-left)? $\cos(x) - \sin(x)$

   (e) What is the Riemann sum approximation? $\sum$ height $\cdot$ base $= \sum \cos(x) - \sin(x)\Delta x$

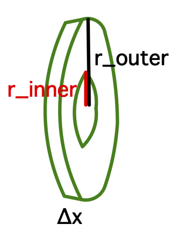
   (f) What are the limits of the integral a and b We are given them from 0 to $\frac{\pi}{4}$.

   (g) Write the integral. Set up but do not solve.

   (h) How could we integrate this? Identify but do not evaluate the integral.
   Calc 1 integrals:
   $$\int_0^{\frac{\pi}{4}} \cos(x) - \sin(x)dx = \sin(x) + \cos(x)\Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1$$

2. For the region from # 1, that is the region between $\sin(x)$ and $\cos(x)$ from 0 to $\frac{\pi}{4}$, find the volume by revolving the same region about the $x-$axis

   (a) Slice perpendicular to this axis of revolution. Sketch a picture of a Riemann slice.



   (b) Which is the infinitesimal part of the slice? Δx

   (c) The slice is an annular region, and $r_{outer} = \cos x$ and $r_{inner} = \sin(x)$ in terms of the slicing variable x.

   (d) The integral is $\int_0^{\frac{\pi}{4}} \pi((\cos x)^2 - (\sin x)^2)dx = \int_0^{\frac{\pi}{4}} \pi(r_{outer}^2 - r_{inner}^2)$height, where dx stands for the height of the annular washer. The idea is that we are taking the volume of the larger cylinder and subtracting the volume of the smaller cylinder.

   (e) How can we integrate this? Identify but do not evaluate the integral.
   We can integrate this—first use the integral of the sum is the sum of the integrals. Then use half-angle trig identities to convert each of them to integrals we can apply $w-$ subs to.

3. A gas station stores its gasoline in a tank under the ground. The tank is a cylinder lying horizontally on its side. The radius of the cylinder is 4 feet, its length is 12 feet, and its top is 10 feet under the ground. Let h be the distance underground. Find the volume of the cylinder via slicing:

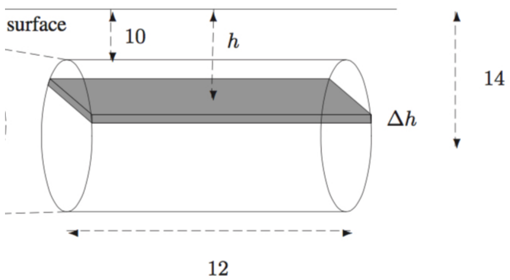
(a) What shape is the slice? rectangle (length·width·height)

(b) Infinitesimal part of the slice? Δh

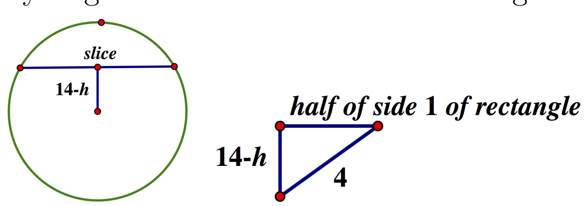
(c) Sketch a diagram and show work to solve for any lengths you need.

h is defined as the distance from ground to a slice so label this on your diagram. Also label 10 feet from ground to the top of the tank, and a 4 foot radius from the top of the tank to the center of the circle.

Next, be sure that you can visualize why 14 is from ground to the center of the circle and why $14 - h$ is the distance from the center of circle to a slice.



(d) Next make a right triangle in this diagram and circle any you use to solve for the slice width: Pythagorean theorem or similar triangles



$$\frac{\text{side 1 rectangle}}{2} = \sqrt{4^2 - (14-h)^2} \text{ so side 1 rectangle} = 2\sqrt{4^2 - (14-h)^2}$$

(e) Any we used: Pythagorean theorem

(f) What is the Riemann sum approximation? We know that side 1 of the rectangle is $2\sqrt{4^2 - (14-h)^2}$, the other side is the length of the cylinder, which is 12, and the height is Δh so we multiply the three to find the volume: $\sum 2\sqrt{4^2 - (14-h)^2}12\Delta h$

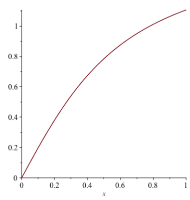
(g) What is a and b? The slices start at $h = 10$ below ground, the top of the tank, and go to $h = 18$ below ground because $18 = 10+$ diameter of the circle, which had a radius of 4. So $a = 10$ and $b = 18$.

(h) Write the integral but do not solve? $\int_{10}^{18} 2\sqrt{4^2 - (14-h)^2}12dh$

(i) What integration methods? trig subs and w-subs

4. For $\arctan(2x)$ from 0 to 1

(a) Sketch a graph over the interval.

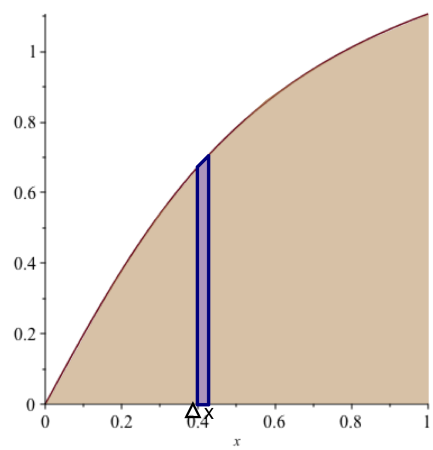


(b) Set up the arc length of the curve.

We use chain rule for the arc length:

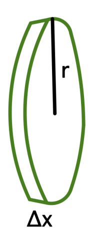
$$\int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+\left[\frac{1}{1+(2x)^2}2\right]^2}dx = \int_0^1 \sqrt{1+\frac{4}{(1+4x^2)^2}}dx.$$

(c) Sketch the region R that gives the area under the curve. Set up the integral that gives the area under the curve and identify geometric/physical components.



$$\int_0^1 \arctan(2x)dx = \int_0^1 \text{height} \times \text{base}$$

(d) Set up the integral that gives the volume by revolving the region R about the $x-$axis and identify geometric/physical components

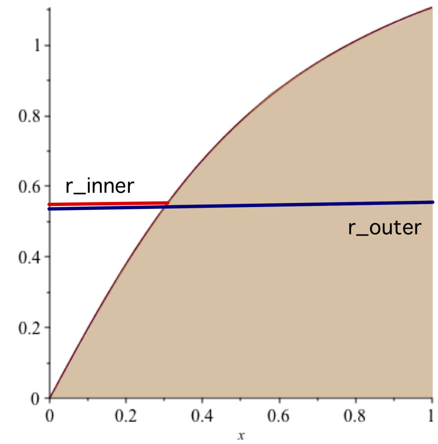


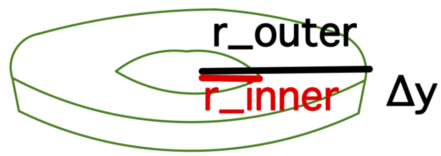
We slice perpendicular to the axis of rotation, i.e. slice in x. We can see from the picture in part c) that the radius goes from the $x-$axis up to the function, so it is $\arctan x$. Thus the volume by rotating the region is

$$\int_0^1 \pi(\arctan(2x))^2dx = \int_0^1 \pi \text{ radius}^2 \text{ height},$$ where dx is the height of the cylindrical disk.

(e) Set up the integral that gives the volume by revolving the region R about the $y-$axis and identify geometric/physical components

We slice perpendicular to the axis of rotation, i.e. slice in y.





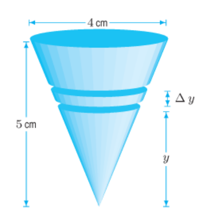
We can see from the picture that the inner radius goes to the function. We express this in the slicing variable y, so we solve $y = \arctan 2x$ for $2x = \tan y$ and $x = \dfrac{\tan y}{2}$. The outer radius goes all the way to $x = 1$ as you can see in the above picture.
Thus the volume by rotating the region is
$\int_0^1 \pi((1)^2 - (\dfrac{\tan y}{2})^2)dy = \int_0^1 \pi(r_{outer}^2 - r_{inner}^2)\text{height}$, where dy stands for the height of the annular washer. The idea is that we are taking the volume of the larger cylinder and subtracting the volume of the smaller cylinder.

(f) Which of the above, if any, can we successfully use w-subs, parts, partial fractions, trig subs, or improper on? Identify but do not evaluate the integrals.

None of our methods apply to part b). Arc length is often very hard to integrate in part b), if not impossible, and this one is no exception. Foiling the denominator will show you none of our methods apply. Numerical methods are useful for the length of the curve.

We can integrate part c) to find the area under the curve by special parts and w-subs.

None of our methods initially apply to part d) nor e). In part e), while none of our 1120 methods initially work, one can use trig identities to convert $(\tan 2x)^2 = (\sec 2x)^2 - 1$ and then use w-subs and Calc 1 integrals from there.

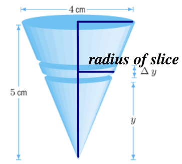


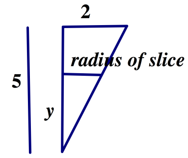
5. Find the volume of the cone standing on its tip via slicing as follows:

(a) What shape is the slice? cylinder/disk ($\pi \cdot \text{radius}^2 \cdot \text{height}$)

(b) Infinitesimal part of the slice? Δy

(c) Sketch a diagram and show work to solve for any lengths you need





$\frac{\text{radius of slice}}{y} = \frac{2}{5}$ so radius $= \frac{2y}{5}$

(d) Any we used: similar triangles

(e) What is the Riemann sum approximation? $\sum \pi (\frac{2y}{5})^2 \Delta y = \sum \pi \frac{4y^2}{25} \Delta y$

(f) What is a and b? The slices start at $y = 0$ and end at $y = 5$, the height of the cone, so that gives us a and b.

(g) Write the integral and solve? $\int_0^5 \pi \frac{4y^2}{25} dy = \frac{4\pi}{25} \int_0^5 y^2 dy = \frac{4\pi}{25} \frac{y^3}{3} \Big|_0^5 = \frac{4\pi}{25} \frac{125}{3} - 0 = \frac{20\pi}{3}$