8.4 Density and 8.5 Work
Group Work Target Practice

1. The density of oil in a circular oil slick on the surface of the ocean at a distance meters from the center of the slick is given by $\delta(r) = \frac{50}{1+r} kg/m^2$. The slick extends from $r = 0$ to $r = 10,000m$. There are many interesting applications, including within what distance half the oil of the slick is contained.

   (a) Slice perpendicular to the density variable and sketch a picture of the resulting slice that has approximately constant density:

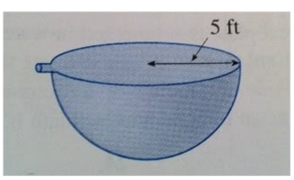
   

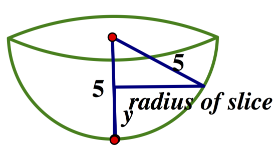
   (b) To find the area of the slice, which of the following should we use?
   $2\pi r\Delta r \approx \pi(r^2_{\text{outer}} - r^2_{\text{inner}})$
   This arises from $\pi(r^2_{\text{outer}} - r^2_{\text{inner}}) = \pi(r + \Delta r)^2 - \pi r^2 = \pi r^2 + 2\pi r\Delta r + \pi\Delta r^2 - \pi r^2 = 2\pi r\Delta r + \pi\Delta r^2$, and when Δr is small Δr^2 goes to 0 faster, so we ignore that term in the limit, and use $2\pi r\Delta r$ for the area as we turn this into an integral

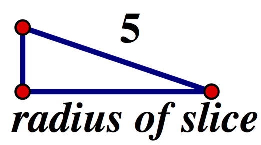
   (c) Find a Riemann sum approximating the total mass of oil in the slick.
   $\sum \delta(r) area \approx \delta(r) 2\pi r\Delta r = \sum \frac{50}{1+r} 2\pi r\Delta r = \sum 100\pi \frac{r}{1+r}\Delta r$

   (d) Find the exact value of the mass of oil in the slick by turning your sum into an integral and evaluating it—look at the r portion and rewrite the numerator as r=(1+r) -1, so that you can break up the integral into two components before you integrate.
   $\int_0^{10000} 100\pi \frac{r}{1+r} dr = \int_0^{10000} 100\pi \frac{(1+r)-1}{1+r} dr = 100\pi \int_0^{10000} \frac{1+r}{1+r} - \frac{1}{1+r} dr$
   $= 100\pi \int_0^{10000} 1 - \frac{1}{1+r} dr = 100\pi(r - ln|1 + r|)\Big|_0^{10000}$
   $= 100\pi(10000 - ln|10001| - 0 + ln|1 + 0|) = 100\pi(10000 - ln|10001|)$

2. A hemispherical tank of radius $5ft$ is filled with gasoline, which weighs $42lb/ft^3$. How much work is required to pump all the fluid to the top rim of the tank? Let y be the height from the bottom of the sphere to a slice.







Work is force times the distance displaced only applies if the force is constant while it is exerted over the distance. So slice the sphere parallel to its base (the force is approximately constant on a slice) and consider how to solve for the volume of the cylindrical slice.

   (a) What is the shape of the slice? cylinder/disk

   (b) hat is the unlabeled height on the pic on the right as a function of y? $5 - y$

   (c) What do we need to solve for the radius of the slice?
   Pythagorean theorem: radius of the slice $= \sqrt{5^2 - (5 - y)^2}$

   (d) What is the volume of the slice? $\pi r^2\Delta y = \pi(5^2 - (5 - y)^2)\Delta y$

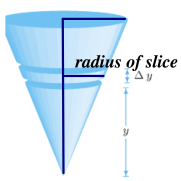
   (e) What is the work required for that slice?
   F d = (volume$\times 42lb/ft^3)\times$ distance the slice must be displaced
   $= (\pi(5^2 - (5 - y)^2)\Delta y \times 42lb/ft^3) \times (5 - y)$

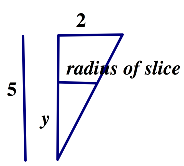
(f) Set up the integral for the total work required.
$\int_0^5 \pi(5^2-(5-y)^2) 42lb/ft^3(5-y)dy$
To evaluate this integral we would expand the quadratic to then use power rule.

3. **Cone (radius 2 ft and height 5 ft) standing on its tip**

(a) What is the volume of one slice of the cone via slicing horizontally. Show reasoning and a picture.
Solution:


cylindrical disk volume $= \pi$ radius of slice$^2 \Delta y$

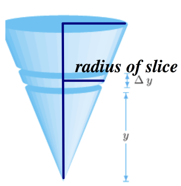
(b) Find the volume of the entire cone via slicing horizontally. Show reasoning and pics.
Solution:
Need to solve for the radius of the slice because we can't have two different variables (r and the y in Δy) in the Riemann sum or the integral.




Similar triangles: $\frac{\text{radius of slice}}{y} = \frac{2}{5}$ so radius $= \frac{2y}{5}$
Sub in: volume of a cylindrical disk slice


$= \pi$ radius of slice$^2 \Delta y = \pi(\frac{2y}{5})^2 \Delta y$
Volume of the entire cone:
y begins at the bottom of the cone at 0, and ends at the top of the cone at 5, so $\int_0^5 \pi(\frac{2y}{5})^2 dy$

(c) If the density $\delta(y)$ of the cone varies with it's height y, what is the total mass. Show reasoning.
Slice perpendicular to y where $\delta(y)$ is approximately constant. Then slicing is as above in this example, so the same process would be followed to solve for the volume.
mass $= \int_0^5 \delta(y)$ volume $= \int_0^5 \delta(y)\pi(\frac{2y}{5})^2 dy$

(d) If the cone is partially filled with fresh water that weighs 62.5 lbs/ft^3 to a height of 4 ft, what is the total work required to pump the water out over the top? Show reasoning and pics.
Work is force times the distance displaced only applies if the force is constant while it is exerted over the distance. So slice the cone parallel to its base (the force is approximately constant on a slice) and consider how to solve for the volume of the cylindrical slice.
F d = ($62.5lb/ft^3 \times$ volume) $\times$ distance the slice must be displaced


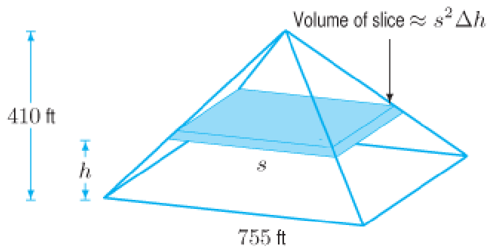
Since the height of the cone is 5, the displacement needed for a slice at height y is $5-y$
F d = ($62.5lb/ft^3 \times$ volume) $\times$ distance the slice must be displaced
= ($62.5lb/ft^3 \times$ volume) $\times(5-y)$
so we must solve for the volume, which we did in part b.

$= (\ 62.5 lb/ft^3 \times \pi(\frac{2y}{5})^2 \Delta y) \times (5-y)$

The only remaining item is to figure out the integration limits. The water begins at the bottom, $y = 0$ and stops at $y = 4$, so those are our limits.

$$\text{Work} = \int_0^4 (62.5 \times (\pi(\frac{2y}{5})^2 \times dy) \times (5-y) = \int_0^4 62.5\pi \frac{4y^2}{25}(5-y)dy$$

4. It is reported that the Great Pyramid of Egypt was built in 20 years. If the stone making up the pyramid has a density of 200 pounds per cubic foot, find the total amount of work done in building the pyramid. The pyramid is 410 feet high and has a square base 755 feet by 755 feet. Assume that the stones were originally located at the approximate height of the construction site and were square slices. Imagine the pyramid was built in layers



   (a) What do we need to solve for $\frac{s}{2}$, with s as in the picture?

   similar triangles: $\frac{\frac{s}{2}}{\frac{755}{2}} = \frac{410-h}{410}$, so $s = \frac{755}{410}(410-h)$

   (b) Set up the integral for the total work required

   Work is force times the distance displaced only applies if the force is constant while it is exerted over the distance. So slice the pyramid parallel to its base (the force is approximately constant on a slice) and consider how to solve for the volume of the rectangular box slice.

   Work to lift a layer F d =

   = (volume$\times 200 lb/ft^3) \times$ distance the slice must be displaced

   $= (s^2 \Delta h 200 lb)(h ft$ displacement)

   $$\text{Total work} = \int_0^{410} (\frac{755}{410}(410-h))^2 dh 200h = \int_0^{410} (\frac{755}{410})^2 200(410-h)^2 h dh$$

   To evaluate this integral we would expand the quadratic and multiple by h, to then use power rule.

5. Medical Case Study: Testing for Kidney Disease

   Patients with kidney disease often have protein in their urine. While small amounts of protein are not very worrisome, more than 1 gram of protein excreted in 24 hours warrants active treatment. The most accurate method for measuring urine protein is to have the patient collect all his or her urine in a container for a full 24 hour period. The total mass of protein can then be found by measuring the volume and protein concentration of the urine. However, this process is not as straightforward as it sounds. Since the urine is collected intermittently throughout the 24 hour period, the first urine voided sits in the container longer than the last urine voided. During this time, the proteins slowly fall to the bottom of the container. Thus, at the end of a 24 hour collection period, there is a higher concentration of protein on the bottom of the container than at the top. One could try to mix the urine so that the protein concentration is more uniform, but this forms bubbles that trap the protein, leading to an underestimate of the total amount excreted. A better way to determine the total protein is to measure the concentration at the top and at the bottom, and then calculate the total protein.

(a) Suppose a patient voids 2 litres (2000 ml) of urine in 24 hours and collects it in a cylindrical container of diameter 10 cm (note that $1cm^3 = 1ml$). A technician determines that the protein concentration at the top is .14 mg/ml, and at the bottom is .96 mg/ml. Assume that the concentration of protein varies linearly from the top to the bottom. Find a formula for the protein concentration in mg/ml, as a function of y, the distance in centimeters from the base of the cylinder.

We are told it is a linear function of y. The $y-$intercept is the concentration where $y = 0$ at the bottom, ie .96. What y value has a concentration of .14? That is the height of the cylinder: 2000 = the volume of the cylinder $= \pi r^2 h = \pi 25h$, so $h = \frac{2000}{25\pi} = 25.46$ Next we must find the slope between the points (0,.96) and $(\frac{2000}{25\pi}, .14)$, which is $\frac{.14-.96}{25.46-0} = -.0322 mg/mlcm$.

So the concentration is .96-.0322y mg/ml

(b) Determine the quantity of protein in a slice of the cylinder.

A cylindrical disk has approximately constant density so

protein = volume times density = $(\pi 25 \Delta y)(.96 - .0322y)$

(c) Write an integral that gives the total quantity of protein in the urine sample. Does this patient require active treatment?

$\int_0^{25.46} \pi 25 dy(.96-.0322y) = \int_0^{25.46} \pi 25(.96-.0322y)dy = 25\pi(.96y-.0322\frac{y^2}{2})\Big|_0^{25.46} = 1099.98mg \approx$ $1.1gm$, so yes, treatment is needed.

6. **Cylinder (radius 4 ft and length 12 ft) laying sideways**

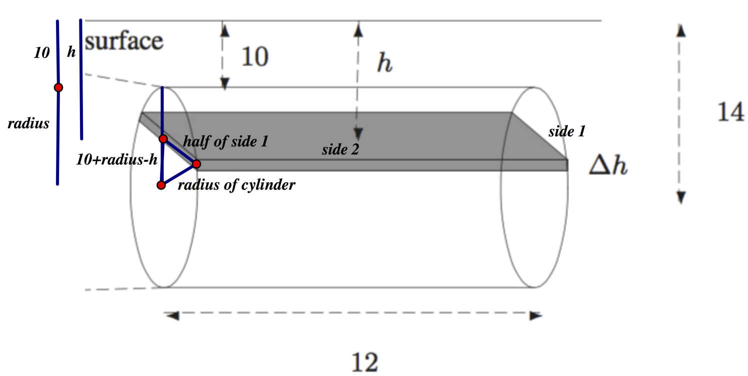
(a) If the cylinder is buried 10 feet under ground, with the height h measured from ground level to a slice, and it is completely filled with salinated water, which weighs 64 lbs/ft^3 (more than fresh water which is 62.5 lbs/ft^3), what is the work required to pump the salinated water above ground? Show reasoning and pics.

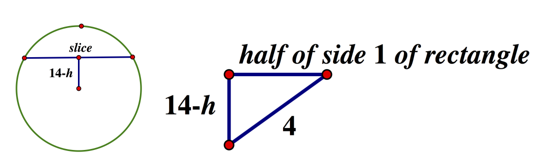
Work is force times the distance displaced only applies if the force is constant while it is exerted over the distance. So slice the cylinder horizontally (the force is approximately constant on a slice) and consider how to solve for the volume of the rectangular box slice.

F d = ($64lb/ft^3 \times$ volume) $\times$ distance the slice must be displaced

= ($64lb/ft^3 \times$ volume) $\times h$

so we must solve for the volume:





The volume of the rectangular box slice is length $\times$ width (side 1) $\times$ height

$$= 12 \times 2\sqrt{4^2 - (14-h)^2} \times \Delta h$$

F d = ($64 lb/ft^3 \times$ volume) $\times$ distance the slice must be displaced

= ($64 lb/ft^3 \times$ volume) $\times h$

= ($64 lb/ft^3 \times \left(12 \times 2\sqrt{4^2 - (14-h)^2} \times \Delta h)\right) \times h$

Add up over the entire water supply. The water begins when h starts at 10 at the top of the tank, and goes to $10 + 2\times$ radius of the cylinder $= 10 + 4 + 4$ to the bottom of the tank.

$$\text{Work} = \int_{10}^{18} (64 \times (12 \times 2\sqrt{4^2 - (14-h)^2} \times dh) \times h = \int_{10}^{18} (64 \times 12 \times 2\sqrt{4^2 - (14-h)^2})h dh$$

(b) If the density $\delta(h)$ of material in a cylinder varies with it's height h, set up the integral that represents the total mass. Show reasoning.

Slice perpendicular to h where $\delta(h)$ is approximately constant. Then slicing is as above in this example, so the same process would be followed to solve for the volume.

mass $= \int_{10}^{18} \delta(h)$ volume $= \int_{10}^{18} \delta(h) 12 \times 2\sqrt{4^2 - (14-h)^2} \times dh$