Numerical Integration Methods in Maple

Instructions: As computer algebra software, Maple can plot and approximate very quickly. Read through the following and execute the command codes by hitting return at the end of the line. Write your responses in your notes.

Goal: Investigate graphical, numerical and algebraic representations of numerical integration, assisted by the computer algebra system Maple.

Definition: An **elementary antiderivative** is one with a closed form involving elementary functions like polynomials, sin, cos, exp...

However there are many integrals that arise in real life that are not elementary, and we approximate them numerically using a variety of methods (like we'll focus on in 7.5, chapter 9 and chapter 10!). Execute:

```
> with(Student[Calculus1]): with(plots):
```

Question 1: Do any of our prior methods (w-subs, parts, partial fractions, trig sub, improper) work to successfully compute an elementary antiderivative for $\int_0^1 e^{-x\sin(x)}\,dx$?

In a situation where Maple cannot find an explicit form for the integral, it will just output the same integral you put in! Execute:

```
> int(exp(-x*sin(x)),x=0..1);
```

That means that Maple can not compute a nice antiderivative.

However, it can approximate the value over a region: `evalf` uses a numerical integration procedure to create the best decimal approximation it knows how to.   Execute:

```
> evalf(int(exp(-x*sin(x)),x=0..1));
```

To do: Write the decimal approximation in your notes as you'll need it later.

We can also approximate integrals in Maple using the `ApproximateInt` command:

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=left, partition=3,
  output = plot);
```

To do: Sketch a rough graph of **left**(3) in your notes as you'll need it later.

Question 2: Compute $\Delta x = \frac{b-a}{n}$, where n is the subintervals used (3), and a=0, b=1 arise from the interval we are computing/plotting on.

Question 3: In this case the left sum (3) will have these terms: $\Delta x f(x_0) + \Delta x\, f(x_1) + \Delta x\, f(x_2)$, where

$x_{i+1} = x_i + \Delta x$

Δx represents the base of a rectangle, and the function value is the height, so we are computing the areas of the rectangles (base times height).

Use the function $f(x)=e^{-x\sin(x)}$ to write out the heights and three terms for the left sum(3) on the interval from 0 to 1.

$\frac{1}{3}e^{-0\sin(0)}+\ldots$

Question 4: Write out the terms for the **right sum** (3)

Question 5: Compute the right sum (3) on your calculator.

Question 6: Execute the following Maple command and compare it with your responses in Questions 4 and 5. Do they match?

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=right, partition=
  3, output = plot);
```

Question 7: The **trapezoid approximation**(3) is the average of the left(3) and the right(3).
What does the sketch for the trapezoid approximation (3) look like for the same function? Use a line that connects the left and right function values to sketch your response in your notes.

Question 8: Compare the plots of left(3) and trap(3) and examine the missing area (the space between the approximation and the curve). Which is a better approximation that minimizes this space?

Question 9: Execute the following and write down the decimal value of the trapezoid approximation.

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=trapezoid,
  partition=3, output = plot);
```

Question 10 (Building Intuition)

If $e^{-x\sin(x)}$ had been purely concave down over the interval, then we could reason that the trapezoid approximation would be an underestimate, because the trapezoid would always be under the curve, like the first trapezoid from 0 to 1/3 in the above Maple plot clearly is.

Yet that doesn't apply here--notice the inflection point change. However, even when concavity changes, we can still examine the spaces between the approximation and the curve geometrically, and reason whether the overall missing space will be above or below the curve.

So examine the missing spaces and see whether there is more space missing above or below the curve--can you visually see that trap(3) is an underestimate of $\int_0^1 e^{-x\sin(x)}\,dx$?

Question 11 Compute the numerical error between Maple's best value and trap(3), ie Maple's best value - trapezoid approximation.
[This is the decimal just after Question 1 - the decimal in Question 9]. The fact that it is positive shows that the trapezoid approximation was an underestimate.

Question 12: Write out the 3 terms for the **midpoint sum** (3), using the midpoints of each interval.

$$\frac{1}{3}e^{-\frac{1}{6}\sin\left(\frac{1}{6}\right)}+$$

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=midpoint,
  partition=3, output = plot);
```

Note that if you compute the error between Maple's best estimate and the midpoint approximation, you'll see that it is smaller than the one in Question 11, and it is negative, indicating first that in this case the midpoint sum(3) is a better estimate, and the negative shows us it is an overestimate (ie overall, there is more missing space above the curve).
Execute the following which computes Maple's best estimate - midpoint approximation:

```
> .7636487064-.7664147547;
```

Simpson's Rule (3) offers the best approximation via

$$\frac{2}{3}\,Midpoint(3)+\frac{1}{3}\,Trapezoid(3)$$

Execute the command to see how nicely it represents the area under the curve and minimizes the missing space:

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=simpson,
  partition=3, output = plot);
```

The 4000 level class "numerical methods" explores in depth a variety of such techniques if you are interested in numerical approximation and errors (like say computer science or mathematics minors or majors might be).

Question 13: We can also get better approximations by increasing the number of subintervals. In the Maple command below for the left sum, modify the "3" next to the **partition=** command to
a) 4 then
b) 5 then
c) 10 then
d) 100 then
e) 1000.

First, Notice that Maple does these very quickly!

Next, which is the best approximation of $\int_0^1 e^{-x\sin(x)}$?

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=left, partition=3,
  output = plot);
```

The visuals show the idea of taking smaller and smaller rectangles as we let $\Delta x = \frac{b-a}{n}$, approach smaller values [example for n=1000 $\Delta x = \frac{1-0}{1000} = .001$]. This is the whole idea of the Riemann sum approximating the integral from Calculus 1: If we look at the limit as Δx goes to 0 (ie to turn Δx to dx) then that is how we compute the integral.

In 8.1 we build upon 7.5 to compute other areas, like areas between curves, using Riemann sum approximation slices that turn into the integral in the limit. Execute the following to help you develop some intuition:

```
> g:= x -> x^2 + 2: f:= x -> 2*x + 5:
  m:= plot(f(x), x = 0..3.1, thickness=2, color = magenta, axes=
  boxed):
  n:= plot(g(x), x = 0..3.1, thickness=2, color = blue,axes=boxed):
  p:= seq( plot([0 + i * (3/100) , t, t = g(0 + i*(3/100))..f(0 + i *
  (3/100))], thickness=2, color=red), i = 0..100):
  display({m,n,p});
```

To do: Draw a rough sketch in your notes.

Question 14: The top curve is $2x+5$ and the bottom curve is $x^2 + 2$, so what is the height, written as a function of x, of one of the Riemann sum rectangles?

Question 15: The width or base of one of the Riemann sum rectangles is Δx, so what is the area (base times height) written as a function of x?

We can symbolically add up all the rectangles by using a summation symbol:
$\sum$ height $\times$ base

To do: Write the Riemann sum approximation in your notes, using the summation symbol and your response to Question 14: Σ Question 14 Δx