Numerical Integration Methods in Maple

Instructions: As computer algebra software, Maple can plot and approximate very quickly. Read through the following and execute the command codes by hitting return at the end of the line. Write your responses in your notes.

Goal: Investigate graphical, numerical and algebraic representations of numerical integration, assisted by the computer algebra system Maple.

Definition: An **elementary antiderivative** is one with a closed form involving elementary functions like polynomials, sin, cos, exp...

However there are many integrals that arise in real life that are not elementary, and we approximate them numerically using a variety of methods (like we'll focus on in 7.5, chapter 9 and chapter 10!). Execute:

```
> with(Student[Calculus1]): with(plots):
```

Question 1: Do any of our prior methods (w-subs, parts, partial fractions, trig sub, improper) work to successfully compute an elementary antiderivative for $\int_0^1 e^{-x\sin(x)}\,dx$?

In a situation where Maple cannot find an explicit form for the integral, it will just output the same integral you put in! Execute:

```
> int(exp(-x*sin(x)),x=0..1);
```

$$\int_0^1 e^{-x\sin(x)}\,dx \qquad (1)$$

That means that Maple can not compute a nice antiderivative.

However, it can approximate the value over a region: `evalf` uses a numerical integration procedure to create the best decimal approximation it knows how to. Execute:

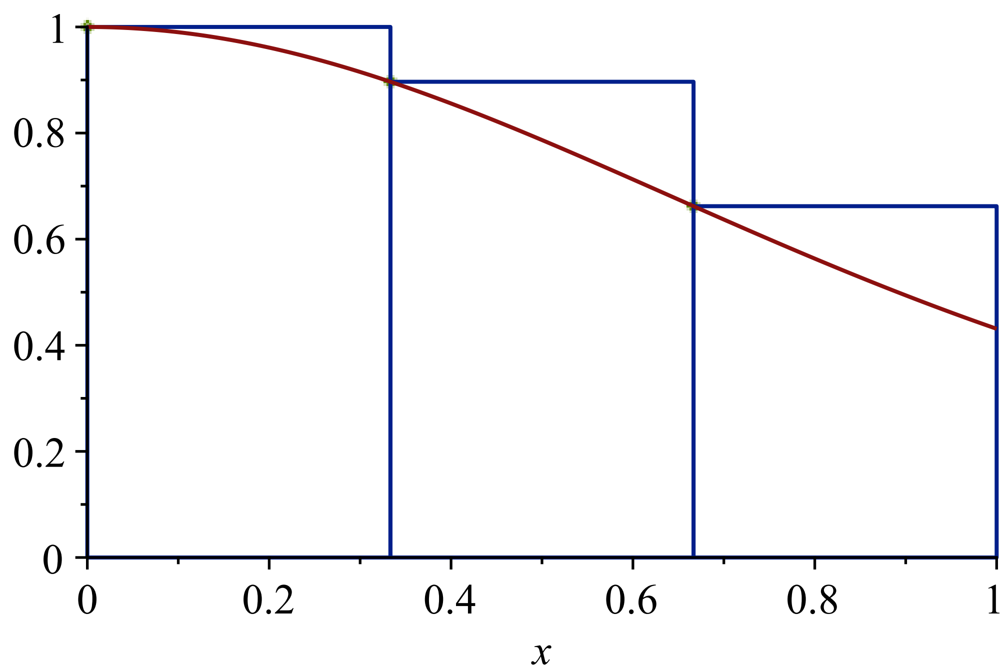
```
> evalf(int(exp(-x*sin(x)),x=0..1));
```

$$0.7636487064 \qquad (2)$$

Question 2: Write the decimal approximation in your notes as you'll need it later.

We can also approximate integrals in Maple using the `ApproximateInt` command:

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=left, partition=3,
  output = plot);
```



A left Riemann sum approximation of $\int_0^1 f(x)\,dx$, where $f(x) = e^{-x\sin(x)}$ and the partition is uniform. The approximate value of the integral is 0.8529444108. Number of subintervals used: 3.

Question 3: Compute $\Delta x = \frac{b-a}{n}$, where n is the subintervals used (3), and a=0, b=1 arise from the interval we are computing/plotting on.

In this case the left sum (3) will have these terms: $\Delta x f(x_0)$+ Δx $f(x_1)$ + Δx $f(x_2)$, where $x_{i+1} = x_i + \Delta x$

Δx represents the base of a rectangle, and the function value is the height, so we are computing the areas of the rectangles (base times height).

We can use the function f(x)=$e^{-x\sin(x)}$ to write out the heights and three terms for the left sum(3) on the interval from 0 to 1:

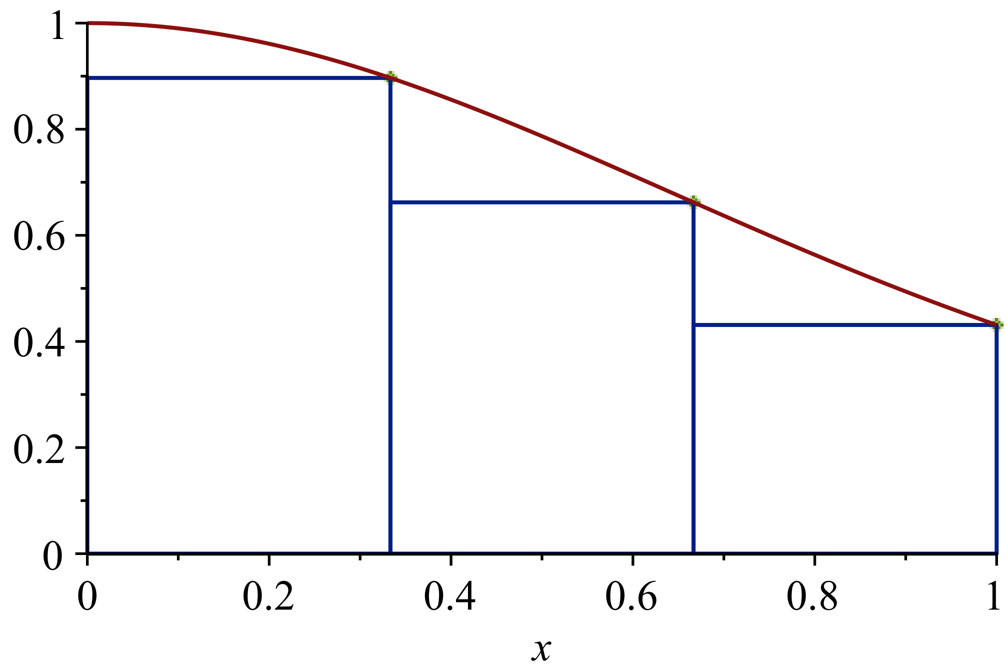
$$\frac{1}{3}e^{-0\sin(0)} + \frac{1}{3}e^{-\frac{1}{3}\sin\left(\frac{1}{3}\right)} + \frac{1}{3}e^{-\frac{2}{3}\sin\left(\frac{2}{3}\right)}$$

Question 4: Write out the terms for the **right sum** (3)

Question 5: Execute the following Maple command and compare it with your responses in Question 4 . Do they match?

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=right, partition=
```

```
3, output = plot); ApproximateInt(exp(-x*sin(x)),x = 0 .. 1,
method=right, partition=3);
```



A right Riemann sum approximation of $\int_0^1 f(x)\, dx$, where $f(x) = e^{-x \sin(x)}$ and the partition is uniform. The approximate value of the integral is 0.6633030610. Number of subintervals used: 3.

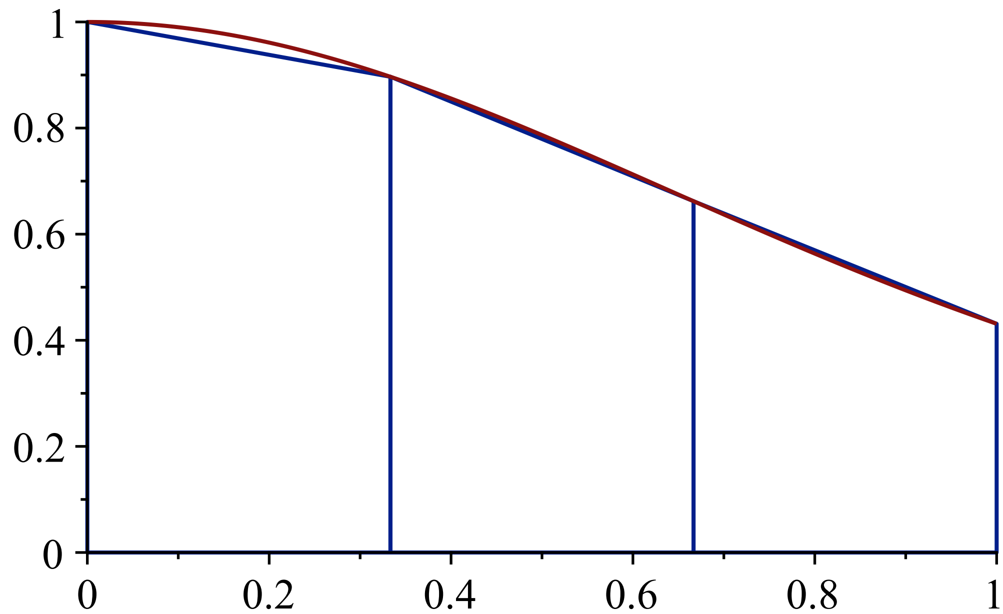
$$\frac{e^{-\frac{\sin\left(\frac{1}{3}\right)}{3}}}{3} + \frac{e^{-\frac{2\sin\left(\frac{2}{3}\right)}{3}}}{3} + \frac{e^{-\sin(1)}}{3} \qquad (3)$$

__Question 6__: The **trapezoid approximation**(3) is the average of the left(3) and the right(3). Is the trapezoid approximation a better approximation than the left sum and the right sum individually?

Yes---if we sketch the trapezoid graph, as Maple does below, we can see that it is a much better fit for the area under the curve:

Next execute the following for the trapezoid approximation.

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=trapezoid,
  partition=3, output = plot);
```



An approximation of $\int_0^1 f(x)\,dx$ using trapezoid rule, where $f(x) = e^{-x\sin(x)}$ and the partition is uniform. The approximate value of the integral is 0.7581237359. Number of subintervals used: 3.

<u>**Question 7**</u>: Write out the 3 terms for the **midpoint sum** (3), using the midpoints of each interval.

$$\frac{1}{3} e^{-\frac{1}{6}\sin\left(\frac{1}{6}\right)} + \frac{1}{3} e^{-\frac{3}{6}\sin\left(\frac{3}{6}\right)} + \frac{1}{3} e^{-\frac{5}{6}\sin\left(\frac{3}{6}\right)}$$

<u>**Question 8**</u>: Execute the following in Maple. Is it the same as your response (note that 3/6=1/2)?

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=midpoint,
  partition=3);
```

$$\frac{e^{-\frac{\sin\left(\frac{1}{6}\right)}{6}}}{3} + \frac{e^{-\frac{\sin\left(\frac{1}{2}\right)}{2}}}{3} + \frac{e^{-\frac{5\sin\left(\frac{5}{6}\right)}{6}}}{3} \qquad \textbf{(4)}$$

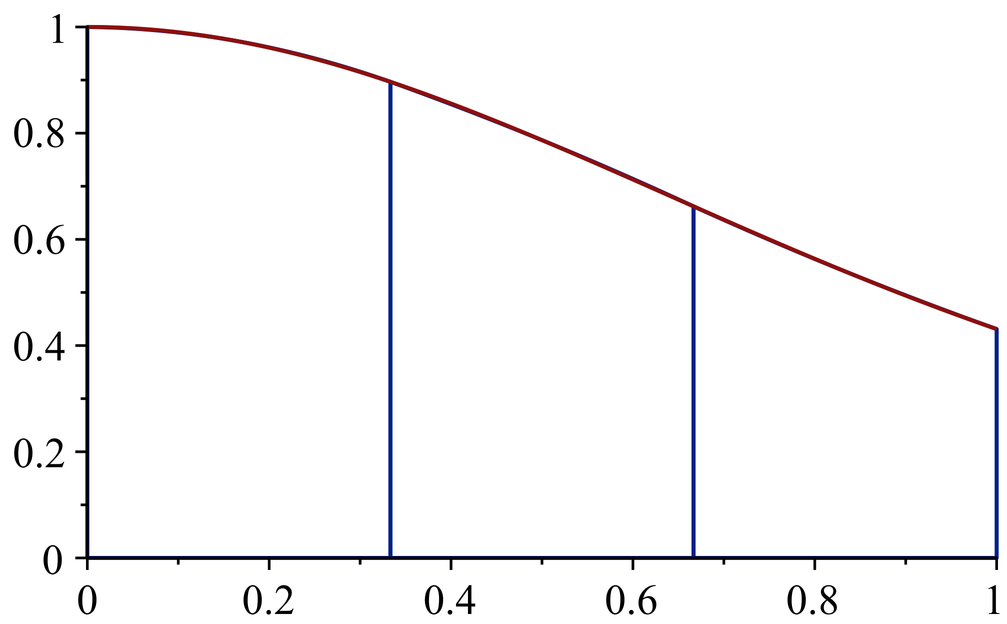
Note that 3/6 = 1/2, so yes they are the same.

Simpson's Rule (3) offers the best approximation via

$$\frac{2}{3}\,\mathit{Midpoint}(3) + \frac{1}{3}\,\mathit{Trapezoid}(3)$$

Execute the command to see how nicely it represents the area under the curve and minimizes the missing space between the approximation and the function:

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=simpson,
  partition=3, output = plot);
```



An approximation of $\int_0^1 f(x)\,dx$ using Simpson's rule, where $f(x) = e^{-x\sin(x)}$ and the partition is uniform. The approximate value of the integral is 0.7636510818. Number of subintervals used: 3.

The 4000 level class "numerical methods" explores in depth a variety of such techniques if you are interested in numerical approximation and errors (like say computer science or mathematics minors or majors might be).

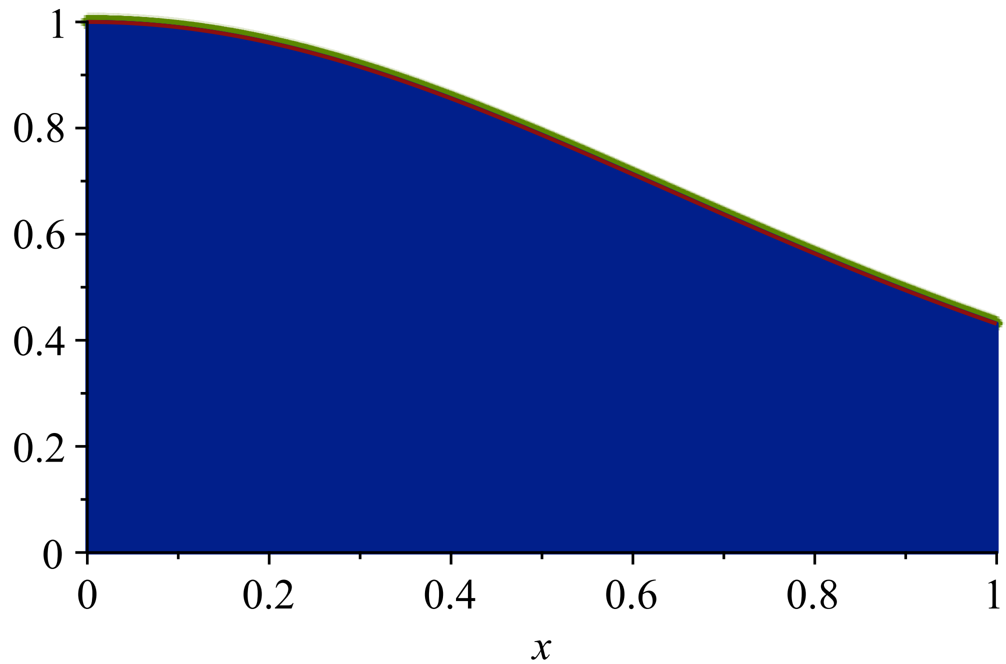
Question 9: We can also get better approximations by increasing the number of subintervals. In the Maple command below for the left sum, modify the "3" next to the **partition=** command to

a) 4 then
b) 5 then
c) 10 then
d) 100 then
e) 1000.

First, Notice that Maple does these very quickly!

Next, of these subintervals, which is the best approximation of $\int_0^1 e^{-x\sin(x)}$? 1000

```
> ApproximateInt(exp(-x*sin(x)),x = 0 .. 1, method=left, partition=
  1000, output = plot);
```

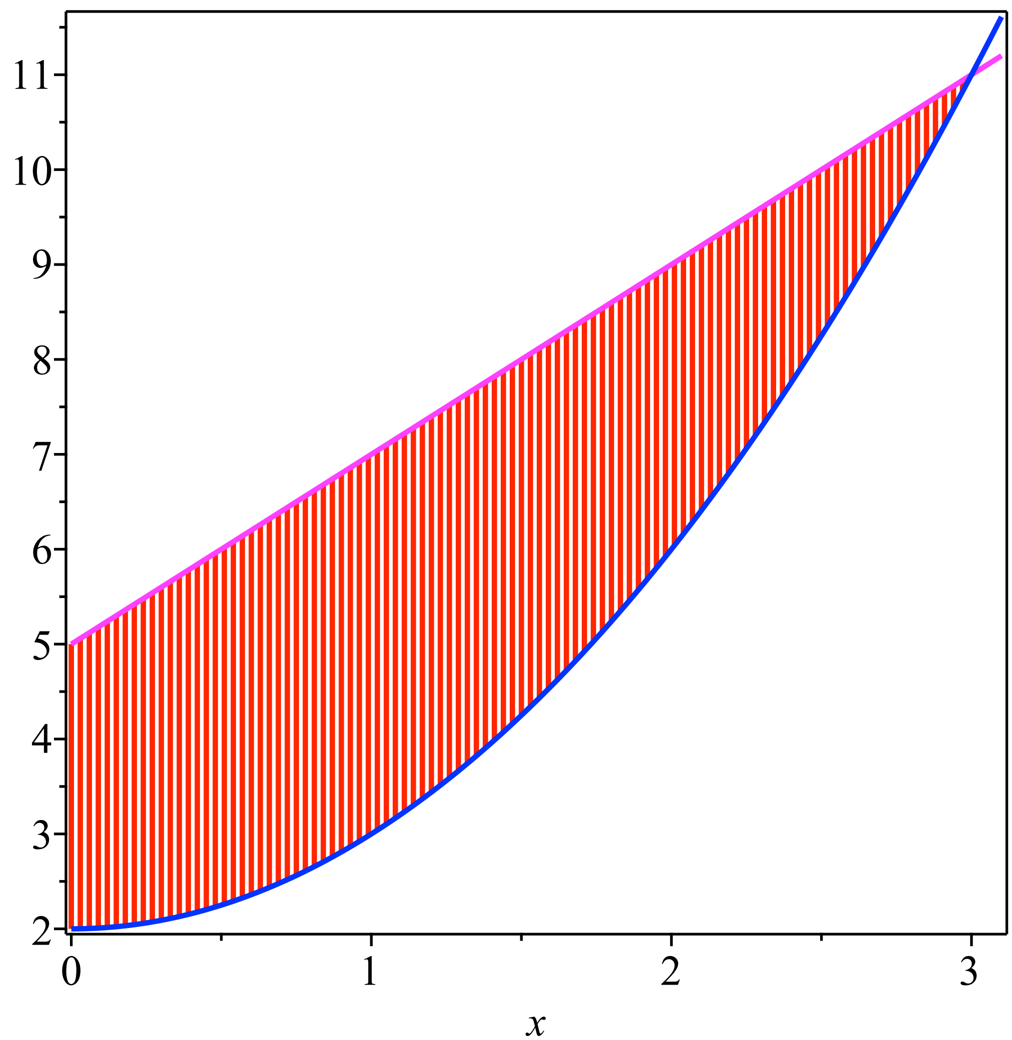


A left Riemann sum approximation of $\int_0^1 f(x)\,dx$, where $f(x) = e^{-x \sin(x)}$ and the partition is uniform. The approximate value of the integral is 0.7639331187. Number of subintervals used: 1000.

The visuals show the idea of taking smaller and smaller rectangles as we let $\Delta x = \frac{b-a}{n}$, approach smaller values [example for n=1000 $\Delta x = \frac{1-0}{1000} = .001$]. This is the whole idea of the Riemann sum approximating the integral from Calculus 1: If we look at the limit as Δx goes to 0 (ie to turn Δx to dx) then that is how we compute the integral.

In 8.1 we build upon 7.5 to compute other areas, like areas between curves, using Riemann sum approximation slices that turn into the integral in the limit. Execute the following to help you develop some intuition:

```
> g:= x -> x^2 + 2: f:= x -> 2*x + 5:
  m:= plot(f(x), x = 0..3.1, thickness=2, color = magenta, axes=
  boxed):
  n:= plot(g(x), x = 0..3.1, thickness=2, color = blue,axes=boxed):
  p:= seq( plot([0 + i * (3/100) , t, t = g(0 + i*(3/100))..f(0 + i *
  (3/100))], thickness=2, color=red), i = 0..100):
  display({m,n,p});
```



To do: Draw a rough sketch in your notes.

Question 10: The top curve is $2x+5$ and the bottom curve is $x^2 + 2$, so what is the height, written as a function of x, of one of the Riemann sum rectangles (top - bottom)? 2x+5 - (x^2 +2)

Question 11: The width or base of one of the Riemann sum rectangles is Δx, so what is the area (base times height, i.e. Δx * Question 10) written as a function of x? Δx* (2x+5 - (x^2 +2))

Note that we can symbolically add up all the rectangles by using a summation symbol:

$\sum$ height × base = $\sum (2x + 5 - (x^\wedge 2 + 2))\Delta x$