Taylor Polynomials and Series in Maple

Goal: Explore Taylor polynomials and series using a variety of representations (numerical, symbolic, and graphical) through pattern exploration assisted by appropriate technology, including the computer algebra system Maple, which is one of the course goals.

Read the text and hit return at the end of each Maple command line (the commands are in red).

Compare your work with others in the class and ask me any questions as I make my way around!

```
> with(Student[Calculus1]): with(plots):
```

What is a Taylor Polynomial/Series?

Polynomials are easy to work with, so we approximate functions with a polynomial.
The first degree Taylor polynomial is the linear approximation of the tangent line that you worked with in Calculus I: $f(a) + f'(a)(x\text{-}a)$.
To get a better match, use more derivatives!

The nth degree Taylor polynomial is $f(a) + f'(a)(x\text{-}a) + \dfrac{f^2(a)}{n!}(x-a)^2 + ... + \dfrac{f^n(a)}{n!}(x-a)^n$

The full Taylor series is a power series centered at a that continues on from $n=0$ to $n=$

∞: $\quad \displaystyle\sum \frac{f^n(a)}{n!}(x-a)^n$

To compute a Taylor polynomial or series, it may be helpful to set up a table as follows:

n	$f^n(x)$	$f^n(a)$	Taylor term $\dfrac{f^n(a)}{n!}(x-a)^n$

For $n=0$, the 0th derivative is the function value, and we evaluate it at the center of the power series.

```
> TaylorApproximation(exp(x), x = 5, order = 0);
```

For $n=1$, $f^n(x)$ is the 1st derivate is the function value, and then we evaluate it at the center of the power series. That gives the degree 1 Taylor term. The full first degree polynomial adds the Taylor terms for $n=$ 0 and $n=1$. It is the linear tangent line approximation:

```
> TaylorApproximation(exp(x), x = 5, order = 1);
```

1) By hand activity: In class we obtained $e^5 + e^5(x-5)$ for the tangent line approximation, now called the degree 1 Taylor polynomial. In your notes, expand the terms and collect like terms to show that this matches Maple's output.

Maple can compute Taylor polynomials of degree n and complete the algebra of collecting like terms very fast!

```
> TaylorApproximation(exp(x), x = 5, order = 50);
```

Where does $n!$ come from?

Maple collects like terms instead of presenting the factorials. Regardless of which representation, the idea is that the polynomial of degree n's nth derivative matches the nth derivative of the function locally, at the center point $x=a$.
As we showed in class, this results from dividing the nth derivative by $n!$ because of the power rule applied to taking n derivatives of the $(x-a)^n$ term (we used both power rule and chain rule in class)

Geometric/graphical representations of the Taylor series

Maple can also output Taylor polynomials as visual representations. Here is the Taylor polynomial of degree 0 (e^5), which is always a horizontal line at the function value at the point, plotted along with the function (e^x).

The sign tells us whether the function is above or below the x axis at a.

```
> TaylorApproximation(exp(x), x = 2, order=0,output=plot);
```

Here is the Taylor polynomial of degree 1 ($e^5 + e^5 \cdot (x - 5)$), which is the linear tangent line approximation, plotted along with the function (e^x). The sign of the Taylor term

$$\frac{f'(a)}{n!}(x - a) = e^5 \cdot (x - 5)$$ tells us whether the function is increasing or decreasing at a.

```
> TaylorApproximation(exp(x), x = 0, order=1,output=plot);
```

The sign of the next Taylor term (degree 2) tells us about concavity.

```
> TaylorApproximation(exp(x), x = 0, order=2,output=plot);
```

2) By hand activity: Using the concavity of the graph above at $x=0$, will the sign of the degree 2 Taylor term be positive or negative? Answer in your notes.

Taylor polynomials for sin(x) about x=0

Notes activity Here $n=15$, and we are computing the Taylor polynomial for sin(x) about $x=0$.

Execute and then write the first few 3 terms in your notes, ie the Taylor polynomial of degree 5:

```
> TaylorApproximation(sin(x), x = 0, order=15);
```

Notice that instead of writing the Taylor series using factorial, like 3!, Maple multiplies out the factorials. Also notice that there are no even terms here, so the degree 4 Taylor polynomial is the same as the degree 3 one--the Taylor term of degree 4 has a 0 coefficient.

Notes activity First plot the linear approximation (Taylor polynomial of degree 1), and then plot of the 15th degree Taylor polynomial approximation, and then sketch them in your notes:

```
> TaylorApproximation(sin(x), x = 0, order=1,output=plot);
```

```
> TaylorApproximation(sin(x), x = 0, order=15,output=plot);
```

Where did *f(x)* go here? Well it is such a good approximation at that point, that the difference between the polynomial and the function is not observable near *x*=0 within this small (local) plotting area. The functions would separate further away from 0:

```
> TaylorApproximation(sin(x), x = 0, -7.5..7.5, order=15,output=plot)
;
```

Class webpage activity: To better process this local to global behavior, go back to the class highlights webpage and examine the Taylor series animation for sin(*x*)

Taylor polynomials for sin(x) about 1.

Want to change what *x=a* is? No problem. Here is the 2nd degree polynomial for sin(*x*) about *x*=1. Write the polynomial in your notes.

```
> TaylorApproximation(sin(x), x = 1, order=2);
```

Where does the 1/2 coefficient of sin(1) come from? Well Maple collects like terms.

3) By hand activity: Write the table in your notes and fill in three rows, for *n*=0, *n*=1, and *n*= 2.

$f^n(x)$ stands for the *n*th derivative of *f(x)* = sin(x), left as a function of *x*, while *a=1* is plugged into $f^n(a)$.

n	$f^n(x)$	$f^n(a)$	Taylor term $\dfrac{f^n(a)}{n!}(x-a)^n$
0			
1			
2			

4) By hand activity: Compute the 2nd degree polynomial for sin(*x*) about *x*=1 by hand in your notes by adding up the Taylor terms.

Note that if we foiled it out, and collected like terms, then we could show that these are the same, and I've done this in solutions.

Notes activity: Let's compare with the plot. Execute and then sketch the degree 2 Taylor polynomial in your notes.

```
> TaylorApproximation(sin(x), x = 1, order=2,output=plot);
```

Taylor polynomials for arctan(*x*) about *x*=0

Notes activity: Sketch this plot in your notes.

```
> TaylorApproximation(arctan(x), x = 0, order=11,output=plot);
```

The degree 11 Taylor polynomial approximates the function very well between -1 and 1. However,

outside of that range it doesn't approximate the function well:

```
> TaylorApproximation(arctan(x), x = 0, -1.3..1.3, order=11,output=
  plot);
```

5) By hand activity: Using the graph above, approximate the error between the Taylor polynomial of degree 11 and the function on the far right side of the graph (x=1.3). To do this, estimate the y-value difference from the tick marks on the graph, and write your estimate in your notes. We'll be computing other errors graphically like we did here, as well as algebraically in 10.4.

Maple activity: Modify the Maple command from above to find the 11th degree polynomial that approximates arctan(x) at x=0.
So you'll need to change the function, the x value, and the order in the command. So modify sin(x), 1, and 2 below:

```
> TaylorApproximation(sin(x), x = 1, order=2);
```

6) By hand activity: Next, using your Maple responses from above, write the general summation form for the Taylor series in your notes, like James Gregory did for this series back in 1671. Start the index at 0.
Hint: n and $(2n+1)$ will both be useful.

$$\sum_{n=0}^{\infty}$$

Using the ratio test on this series representation, we can compute the radius of convergence, just like we did in 9.5 on power series, since a Taylor series is a power series:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} \frac{1}{2 \cdot (n+1)+1} x^{2 \cdot (n+1)+1}}{(-1)^n \frac{1}{2n+1} x^{2n+1}} \right| = \lim_{n \to \infty} \left| \frac{\frac{1}{2n+2+1} x^{2n+2+1}}{\frac{1}{2n+1} x^{2n+1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\frac{1}{2n+3} x^{2n+3}}{\frac{1}{2n+1} x^{2n+1}} \right| = \lim_{n \to \infty} \left| \frac{2n+1}{2n+3} x^{2n+3-(2n+1)} \right| = \lim_{n \to \infty}$$

$$\left| \frac{2n+1}{2n+3} x^{2n+3-(2n+1)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2n+1}{2n+3} x^{2n+3-2n-1} \right| = \lim_{n \to \infty} \left| \frac{2n+1}{2n+3} x^2 \right| = \lim_{n \to \infty} \frac{2n+1}{2n+3} x^2$$

Since this approaches infinity over infinity, we use L'Hopitals to continue, by taking the derivative of the numerator divided by the derivative of the denominator, with respect to n

$$= \lim_{n \to \infty} \frac{2}{2} x^2 = x^2$$

The ratio test converges when L < 1. Here this means that $L = x^2 < 1$.
So $-1 < x^2 < 1$. This tells us that $-1 < x < 1$. Thus the radius of convergence $r = 1$.

Ask me any questions on the above steps and show me your paper responses as you continue to:

Class webpage activity: Go back to the class highlights webpage and examine the Taylor series animation for arctan(x) to better appreciate the radius of convergence.