

Taylor Polynomial Error Bounds in Maple

Goal: Explore Taylor polynomials and series using a variety of representations (numerical, symbolic, and graphical) through pattern exploration assisted by appropriate technology, including the computer algebra system Maple, which is one of the course goals.

One fundamental question is how well a Taylor polynomial approximates the function.

We can determine the interval of convergence for a Taylor series by geometric series or ratio test techniques, like in 9.5, or if it inherits an interval of convergence from a known series that it is derived from, like in 10.3.

In 10.4 we work with an error bound. Sometimes referred to as Taylor's theorem or Taylor's inequality, named for Brook Taylor who investigated the asymptotic behavior of how well Taylor polynomials fit, it was Joseph-Louis Lagrange who first stated the explicit version in 10.4 that is also known as the Lagrange error bound. Taylor's theorem is an important technique in mathematical analysis and computer science and is also used in mathematical physics and other fields.

Read the text and hit return at the end of each Maple command line (the commands are in red).

Compare your work with others in the class and ask me any questions as I make my way around!

> with(Student[Calculus1]): with(plots):

1) Computing the polynomial by-hand:

Compute the 2nd degree polynomial for $\ln(x)$ about $x=4$ by hand in your notes by computing as in the table below and then adding up the Taylor terms.

Write the table in your notes and fill in three rows, for $n=0$, $n=1$, and $n=2$.

$f^n(x)$ stands for the n th derivative of $f(x) = \ln(x)$, left as a function of x , while $a=4$ is plugged into $f^n(a)$.

n	$f^n(x)$	$f^n(a)$	Taylor term $\frac{f^n(a)}{n!} (x - a)^n$
0	$\ln(x)$	$\ln(4)$	$\frac{\ln(4)}{0!} (x - 4)^0 = \ln(4)$
1	$\frac{1}{x} = x^{-1}$	$\frac{1}{4}$	$\frac{1}{1!} (x - 4)^1 = \frac{1}{4} (x - 4)$
2	$\frac{-1}{x^2} = -x^{-2}$	$\frac{-1}{4^2}$	$\frac{-1}{2!} (x - 4)^2 = \frac{-4^{-2}}{2!} (x - 4)^2$

Then add up the Taylor terms. Compare with the Taylor series statement on the Series Theorems sheet.

$$\ln(4) + \frac{1}{4}(x-4) + \frac{-1}{2!} \frac{4^2}{(x-4)^2}$$

2) Comparison activities: Execute the following Maple command and compare the degree 0 term from your table with Maple's output.

$$> \text{TaylorApproximation}(\ln(x), x = 4, \text{order} = 0);$$

$$2 \ln(2) \tag{1}$$

After comparing the degree 0 terms, notice how they differ and figure out and write down in your notes what algebraic property of logarithms is Maple using from the Algebra, Geometry, Trigonometry and Derivative Review sheet?

$$\ln(4) = \ln(2^2) = 2 \ln(2) \text{ so we see that Maple is using } \ln(a^b) = b \ln(a)$$

Next execute the following degree 2 polynomial:

$$> \text{TaylorApproximation}(\ln(x), x = 4, \text{order} = 2);$$

$$2 \ln(2) + \frac{x}{2} - \frac{3}{2} - \frac{x^2}{32} \tag{2}$$

Then examine the Maple (input) command below (in red) to check your by-hand work for the Taylor polynomial from 1).

$$> \text{expand}(\ln(4) + 1/4/1!*(x-4)^1 - 4^{(-2)}/2!*(x-4)^2);$$

$$2 \ln(2) + \frac{x}{2} - \frac{3}{2} - \frac{x^2}{32} \tag{3}$$

One you execute the above command we see Maple foils the degree one and two terms to expand them, and collects like terms. The table computations in 1) and Maple's outputs are the same.

3) Using the polynomial:

Use your polynomial to approximate $\ln(4.2)$ by plugging in 4.2 for x and computing the resulting value on your calculator. Write the result in your notes.

$$\ln(4) + \frac{1}{4}(4.2 - 4) + \frac{-1}{2!} \frac{4^2}{(4.2 - 4)^2} = \ln(4) + \frac{1}{4}(.2) + \frac{-1}{2!} (.2)^2$$

$$> \text{evalf}(\text{subs}(x=4.2, \ln(4) + 1/4/1!*(x-4)^1 - 4^{(-2)}/2!*(x-4)^2));$$

$$1.435044361 \tag{4}$$

Error Bounds

Next we'll examine error in a number of different ways. Recall that the idea is that polynomials are easier to work with in general, so if we can approximate a function with a polynomial that is a good match, this is helpful in high speed computations and analytic applications. Even though we don't see the full power of these applications in the Calculus II and Analytic Geometry setting, we can explore many of the important ideas that relate.

4) Finding the error in using the polynomial at $x=4.2$

Compute the actual value of the function $f(4.2) = \ln(4.2)$ on your calculator and then compute the absolute value of the difference between the function value and the degree 2 Taylor polynomial from 3) above by

subtracting: $|f(4.2) - P_2(4.2)| = |\ln(4.2) - \text{degree 2 polynomial}(4.2 \text{ from } 3)|$, and write both of these in your notes.

```
> evalf(ln(4.2));
```

1.435084525 (5)

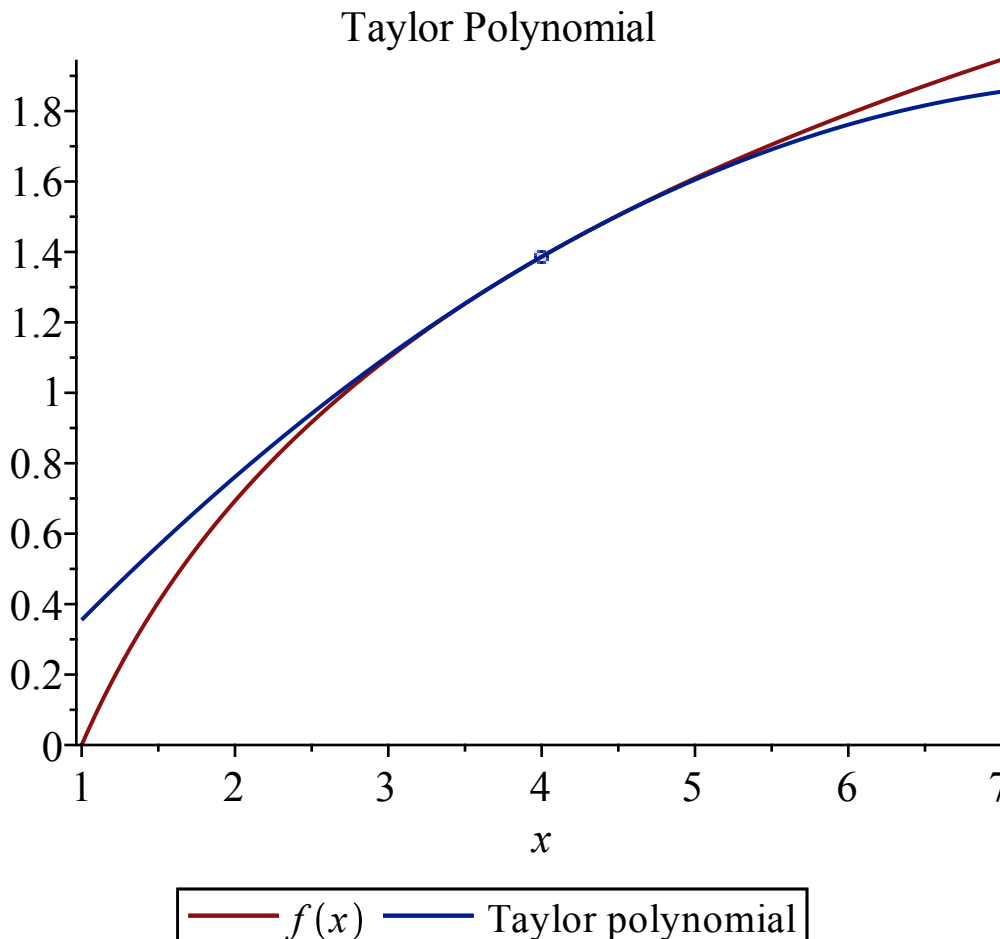
```
> 1.435084525 - 1.435044361;
```

0.000040164 (6)

5) Finding the error graphically over an interval

Examine the following graphs of $\ln(x)$ and the degree 2 Taylor polynomial over the interval from 1 to 7.

```
> TaylorApproximation(ln(x), x = 4, 1..7, order = 2, output=plot);
```



At $x = 4$, for the function $f(x) = \ln(x)$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 2.

First notice that at $x=4.2$ the difference between the graph isn't even visible to the eye because it is a very good fit in this case, at least for value close to the center of the power series.

Over the entire interval $[1, 7]$, we can use the largest y -value difference between the function and the Taylor polynomial on an interval to bound the error by visually inspecting it and then using the tick marks to estimate the error. In your notes, sketch the graphs, label the largest y -value difference in the graph, and write down an estimate for the error on the interval.

We can see that the largest y -value difference occurs at $x=1$.

To compute an error estimate graphically, we can look at the value of the polynomial (close to .35 at $x=1$) and subtract the value of the function (close to 0 at $x=1$) and is about .35.

6) Lagrange Error Bound/Taylor's Inequality Example 1:

Read through the Taylor polynomial error on the on the Series Theorems sheet and write down any questions you have on it (if any) to ask me.

Find the Lagrange error bound for the degree 2 polynomial for $\ln(x)$ about 4 on the interval $[4, 4.2]$ as follows:

Respond in your notes.

a) What is $n+1$?

$$2+1 = 3$$

b) Compute the $(n + 1)$ st derivative of $f(x)$ --you already computed the 2nd derivative in the Table in 1) so make use of that.

The second derivative was $\frac{-1}{x^2} = -x^{-2}$ and we need the third derivative $- -2 x^{-3} = 2 x^{-3}$

c) Write the absolute value of the $(n + 1)$ st derivative of $f(x)$.

It already is positive $2 x^{-3}$

d) Is the absolute value of the $(n + 1)$ st derivative increasing or decreasing on the interval $[4, 4.2]$ between 4 and 4.2?

$2 x^{-3} = \frac{2}{x^3}$ is decreasing as x gets larger

e) Find M as a natural upper bound on the absolute value of the $(n + 1)$ st derivative on the interval $[4, 4.2]$ and write down the bound and your reasoning.

Since the absolute value of the 3rd derivative is decreasing, then it achieves its maximum on the left endpoint, at $x=4$. So a natural upper bound is $M = \frac{2}{4^3}$

f) In your notes, fill in the various components from the Taylor polynomial error on the Series Theorems sheet, but do not simplify.

$$\left| \frac{M}{(n+1)!} (x-a)^{n+1} \right| = \left| \frac{\frac{2}{4^3}}{(2+1)!} (4.2-4)^{2+1} \right|$$

g) Compute the Lagrange error bound/Taylor inequality on your calculator and write it in your notes.

0.00004166666667

The Maple command for this is below.

h) Compare to 4) above. Was 4) within the error predicted by 6 g)?
 4) had 0.000040164 and
 6g) had 0.00004166666667
 so yes, 4) was within the error predicted by 6g)

```
> evalf(2/(4^3)/3!*(.2)^3);
0.00004166666667 (7)
```

7) Lagrange Error Bound/Taylor's Inequality Example 2:

Find the Lagrange error bound for the linear approximation (the degree 1 Taylor polynomial) for $f(x) = \frac{1}{1-x}$ about 0 on the interval $[0, 0.1]$ as follows:

Respond in your notes.

a) What is $n+1$?

$$1+1=2$$

b) Compute the $(n+1)$ st derivative of $f(x)$.

$$f(x) = \frac{1}{1-x} = (1-x)^{-2} \text{ so}$$

$$f^{(1)}(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2} \text{ [chain rule and then the 2 negatives cancel].}$$

$$\text{Similarly, } f^{(2)}(x) = -2(1-x)^{-3} \cdot (-1) = 2(1-x)^{-3}$$

c) Write the absolute value of the $(n+1)$ st derivative of $f(x)$.

$$|2(1-x)^{-3}|$$

d) Is the absolute value of the $(n+1)$ st derivative increasing or decreasing on the interval $[0, 0.1]$?

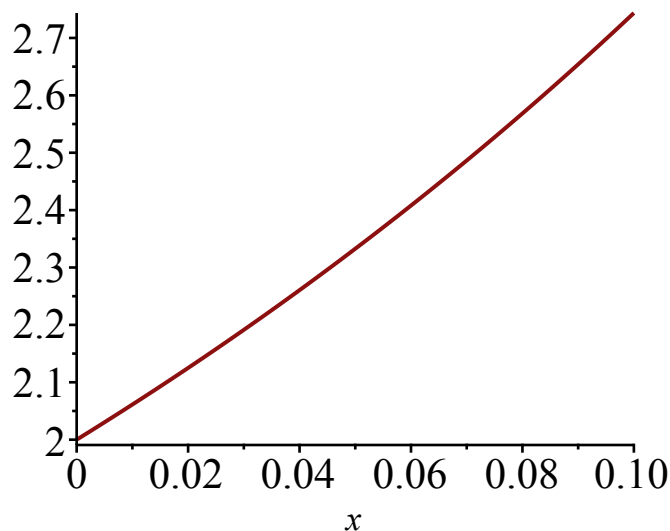
One way to check would be to plug in 0 and 1 and see which is larger.

$$2(1-0)^{-3} = 2$$

$$(1-.1)^{-3} \sim 2.743484224$$

Another is to reason via the function to see that it is increasing. As $0 < x < 1$ gets larger, $1-x$ gets smaller, and the reciprocal times 2 gets larger. Here is a plot

$$\text{plot}(2(1-x)^{-3}, x=0..1);$$



e) Find M as a natural upper bound on the absolute value of the $(n + 1)$ st derivative on the interval $[0, 0.1]$ and write down the bound and your reasoning.

Since the function is increasing the largest absolute value of the $(n + 1)$ st derivative on the interval $[0, 0.1]$ will be at the right endpoint, at .1, so $2(1 - .1)^{-3} \sim 2.743484224$

f) In your notes, fill in the various components from the Taylor polynomial error on the Series Theorems sheet, but do not simplify.

$$\left| \frac{M}{(n + 1)!} (x - a)^{n + 1} \right| = \left| \frac{2(1 - .1)^{-3}}{(1 + 1)!} (.1 - 0)^{1 + 1} \right|$$

Note that $\frac{1}{1 - x}$ arises from geometric power series, so the linear approximation is $1 + x$, and the bound

we have here is the largest error we would obtain by using $1 + x$ to estimate $\frac{1}{1 - x}$

on the interval $[0, .1]$.