Taylor Polynomials and Series in Maple

Goal: Explore Taylor polynomials and series using a variety of representations (numerical, symbolic, and graphical) through pattern exploration assisted by appropriate technology, including the computer algebra system Maple, which is one of the course goals.

Read the text and hit return at the end of each Maple command line (the commands are in red).

Compare your work with others in the class and ask me any questions as I make my way around!

```
> with(Student[Calculus1]): with(plots):
```

What is a Taylor Polynomial/Series?

Polynomials are easy to work with, so we approximate functions with a polynomial.
The first degree Taylor polynomial is the linear approximation of the tangent line that you worked with in Calculus I: *f(a) + f'(a)(x-a)*.
To get a better match, use more derivatives!

The *n*th degree Taylor polynomial is *f(a) + f'(a)(x-a)* + $\frac{f^2(a)}{n!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$

The full Taylor series is a power series centered at *a* that continues on from *n*=0 to *n*= ∞: $\sum \frac{f^n(a)}{n!}(x-a)^n$

To compute a Taylor polynomial or series, it may be helpful to set up a table as follows:

n $\quad f^n(x) \quad f^n(a)$ Taylor term $\frac{f^n(a)}{n!}(x-a)^n$

For *n*=0, the 0th derivative is the function value, and we evaluate it at the center of the power series.

```
> TaylorApproximation(exp(x), x = 5, order = 0);
```

$$e^5 \tag{1}$$

For *n*=1, $f^n(x)$ is the 1st derivate is the function value, and then we evaluate it at the center of the power series. That gives the degree 1 Taylor term. The full first degree polynomial adds the Taylor terms for *n*= 0 and *n*=1. It is the linear tangent line approximation:

```
> TaylorApproximation(exp(x), x = 5, order = 1);
```

$$e^5 x - 4 e^5 \tag{2}$$

<u>1) By hand activity</u>: In class we obtained $e^5 + e^5 (x-5)$ for the tangent line approximation, now called the degree 1 Taylor polynomial. In your notes, expand the terms and collect like terms to show that this matches Maple's output.

In class we obtained $e^5 + e^5 (x-5)$. Expand the terms and collect like terms to show that this matches Maple.

Yes it is the same! Expand $e^5 + e^5 (x-5)$

$= e^5 + e^5 x - 5e^5$

$= e^5 x + e^5 - 5e^5$

$= e^5 x - 4e^5$

which is the Maple output

Maple can compute Taylor polynomials of degree n and complete the algebra of collecting like terms very fast!

```
> TaylorApproximation(exp(x), x = 5, order = 50);
```

$$\frac{19865531204349908714282210763299\,e^{5}\,x^{14}}{257028300224172538893154933081027864480972800} \tag{3}$$

$$+\frac{2967662042789819659748397221706040 3\,e^{5}\,x^{13}}{274262965712418393956437018326014643608223744 00}$$

$$+\frac{90216926100810526969577021694648781 37\,e^{5}\,x^{12}}{641353396742886090482745027470065012745384755200}$$

$$+\frac{2513185798522578932319553653798126208 7\,e^{5}\,x^{11}}{1488856099581699852906372385198365208158928896 00}$$

$$+\frac{69158921236984973190528574086763978 67\,e^{5}\,x^{10}}{372463442379161008544815070677276978587893760 0}$$

$$+\frac{12544075203060524428233909479695226990 87\,e^{5}\,x^{9}}{67557581119359345104210098743388271105393295360}$$

$$+\frac{131000828607096936194216160669761365506089\,e^{5}\,x^{8}}{78391139172734086931732078541877561426798706688 0}$$

$$+\frac{1042111591569456127422079175082278292239609 87\,e^{5}\,x^{7}}{779501890148874576927410856000795001437729639628 80}$$

$$+\frac{48266221083216915375353721704051170221420499 9\,e^{5}\,x^{6}}{515760649121059870448211694196014737793385024716 80}$$

$$+\frac{51584523782688078307407471081801142317667329 19\,e^{5}\,x^{5}}{918698656246887894235877080286651251694467075276 80}$$

$$+\frac{54237442148654893763217206050511320591887458128 7\,e^{5}\,x^{4}}{193189203142202711473601580311707234642036504972 4928}$$

$$+\frac{892205923345373002404922675733030514565253520201 49\,e^{5}\,x^{3}}{794490597922308650935186499031896002465375126699 37664}$$

$$+\frac{856517686411558082308725805083497364899772895994 9929\,e^{5}\,x^{2}}{254236991335138768299259679690206720788920040543 8005248}$$

$$+\frac{891189802639581008584745757972 91\,e^{5}\,x^{15}}{17295862702584943763018550705244166714032128000}$$

$$+\frac{25462565789703378899649439346101\,e^{5}\,x^{16}}{790668009261026000595133746525447621212897280 00}$$

$$+\frac{1872247484536465317577390846 91\,e^{5}\,x^{17}}{988335011576282500743917183156809526516121600 0}$$

$$+\frac{463140998030968610482585717\,e^{5}\,x^{18}}{44007495877237068678764401476369997470105600 0}$$

$$+\frac{3545923266128037295276522297\,\mathrm{e}^{5}\,x^{19}}{640171541589182983436400902726569806947942400}$$

$$+\frac{2287692429952297109131 2749\,\mathrm{e}^{5}\,x^{20}}{8260277955989457850792269712600900734812160000}$$

$$+\frac{54159385150573016421 01\,\mathrm{e}^{5}\,x^{21}}{410667227925612251104729318098056144486400 00}$$

$$+\frac{131476576776595493486801\,\mathrm{e}^{5}\,x^{22}}{219324621590064915348622333748368743648460 80000}$$

$$+\frac{234779597129303040127 1\,\mathrm{e}^{5}\,x^{23}}{900797552959195188038984585037943054270464 0000}$$

$$+\frac{173910830349008111321\,\mathrm{e}^{5}\,x^{24}}{16014178719274581120693059289563432075919360000}$$

$$+\frac{12573072222496853\,\mathrm{e}^{5}\,x^{25}}{2894407663258129901802533850774188851200000 0}$$

$$+\frac{66888934958546 53921\,\mathrm{e}^{5}\,x^{26}}{400354467981864528017326482239085801897984000000}$$

$$+\frac{1393499610088127 77\,\mathrm{e}^{5}\,x^{27}}{225199388239798797009746146259485763567616000000}$$

$$+\frac{1211821719582407 3\,\mathrm{e}^{5}\,x^{28}}{548311553975162288371555834370921859121152000000}$$

$$+\frac{68831832378173\,\mathrm{e}^{5}\,x^{29}}{90346790143634695243040449981572351787008000000}$$

$$+\frac{26257980879929\,\mathrm{e}^{5}\,x^{30}}{1032534744498682231349033714075112591851520000000}$$

$$+\frac{191951419039\,\mathrm{e}^{5}\,x^{31}}{235357184407787861557500331884768311377920000000}$$

$$+\frac{198488399\,\mathrm{e}^{5}\,x^{32}}{76099918760044338130274409667635865190400000000}$$

$$+\frac{234653053\,\mathrm{e}^{5}\,x^{33}}{325110201176408717349972757617371554775040000000}$$

$$+\frac{1408067509\,\mathrm{e}^{5}\,x^{34}}{4941675057881412503719585915784047632580608000000 0}$$

$$+\frac{5855137\,\mathrm{e}^{5}\,x^{35}}{54049570945577949259432970953888020981350400000000}$$

$$+\frac{85283633\,\mathrm{e}^{5}\,x^{36}}{1297189702693870782226391302893312503552409600000000}$$

$$
\begin{aligned}
&- \frac{709183\, e^{5}\, x^{37}}{214267941962826870278466420567198940318924800000000} \\
&+ \frac{2883169\, e^{5}\, x^{38}}{10021146824107595163792891054219765824146636800000000} \\
&- \frac{40967\, e^{5}\, x^{39}}{2326337655596406020166206851872445637748326400000000} \\
&+ \frac{125417\, e^{5}\, x^{40}}{11843173519399885193573416700441541428536934400000000} \\
&- \frac{33151\, e^{5}\, x^{41}}{60696264286924411617063760589762899821251788800000000} \\
&+ \frac{28669\, e^{5}\, x^{42}}{1132996933355922350185190197675574129996700057600000000} \\
&- \frac{281\, e^{5}\, x^{43}}{276811750763094665102063514204827770396921036800000000} \\
&+ \frac{71\, e^{5}\, x^{44}}{2014690035629140118938326930453182569505410252800000000} \\
&- \frac{e^{5}\, x^{45}}{969909898909298874826728336451278770792300544000000000} \\
&+ \frac{47\, e^{5}\, x^{46}}{1886613311935573354234390432731515963301137743872000000000} \\
&- \frac{e^{5}\, x^{47}}{2096945201441904167375386663407664702452953776128000000000} \\
&+ \frac{e^{5}\, x^{48}}{14604606579454203142190928291027500045319395711385600000000000} \\
&- \frac{e^{5}\, x^{49}}{152070466008566890218063040830323844221888207844802560000000000} \\
&+ \frac{e^{5}\, x^{50}}{3041409320171337804361260816606476884437764156896051200000000000} \\
&+ \frac{1049234165854158650828189106679810763137580608018670990\, e^{5}\, x}{155720157192772495583296553810251616483213524833077821440} \\
&+ \frac{8393873326833269206625512857076464912192357815809524170\, e^{5}}{124576125754217996466637243048201293186570819866462257152}
\end{aligned}
$$

Here is the degree 2 Taylor polynomial:

```
> TaylorApproximation(exp(x), x = 5, order = 2);
```

$$
\frac{17\, e^{5}}{2} - 4\, e^{5}\, x + \frac{e^{5}\, x^{2}}{2} \qquad \textbf{(4)}
$$

In class we obtained $e^5 + e^5 (x-5) + \frac{e^5}{2!}(x-5)^2$. Expand the terms and collect like terms to show that this matches Maple.

Yes it is the same! Expand $e^5 + e^5 (x-5) + \frac{e^5}{2!}(x-5)^2$

$= e^5 + e^5 x - 5e^5 + \frac{e^5}{2!}\left(x^2 - 10x + 25\right)$

$= e^5 + e^5 x - 5e^5 + \frac{e^5}{2!}\left(x^2 - 10x + 25\right)$

$= e^5 + e^5 x - 5e^5 + \frac{e^5}{2}x^2 - 5e^5 x + \frac{25}{2}e^5$

$= \left(e^5 - 5e^5 + \frac{25}{2}e^5\right) + \left(e^5 x - 5e^5 x\right) + \frac{e^5}{2}x^2$

$= \frac{17}{2}e^5 - 4e^5 x + \frac{e^5}{2}x^2$

which is the Maple output of the degree 2 Taylor polynomial.

Where does *n!* come from?

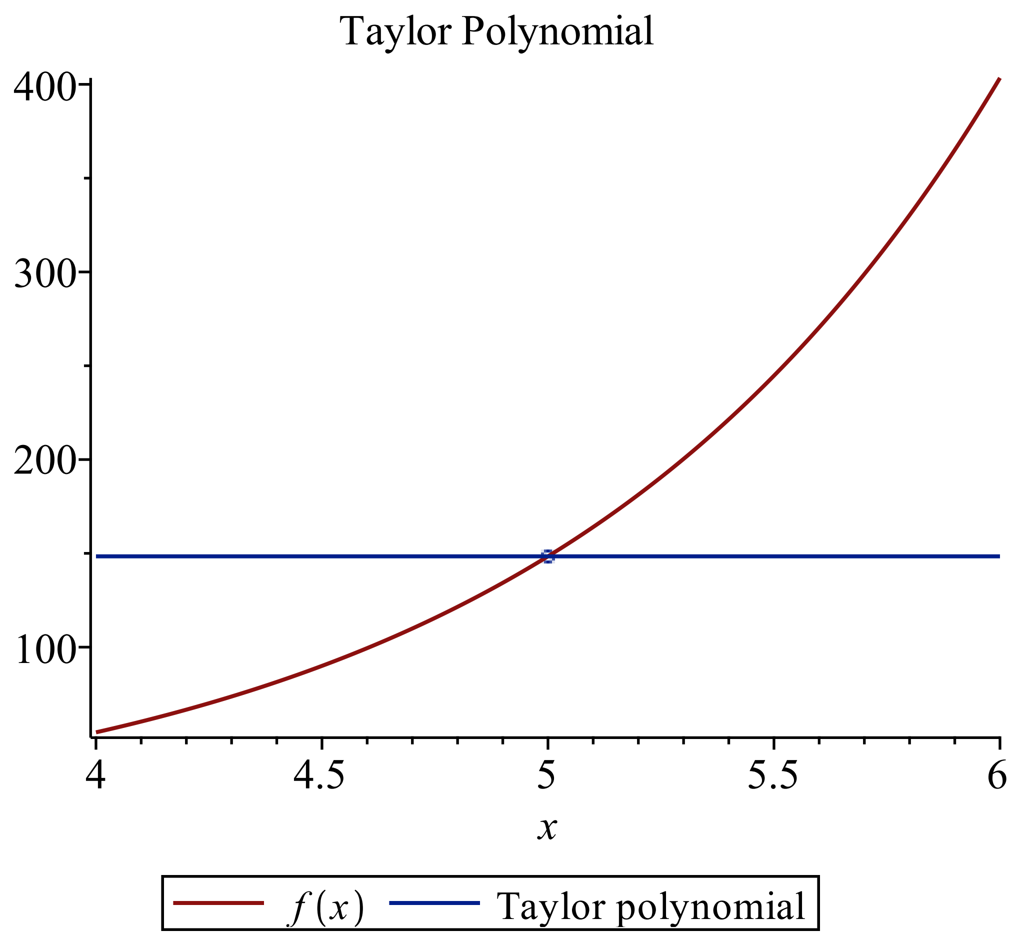
Maple collects like terms instead of presenting the factorials. Regardless of which representation, the idea is that the polynomial of degree n's nth derivative matches the nth derivative of the function locally, at the center point $x=a$.
As we showed in class, this results from dividing the nth derivative by $n!$ because of the power rule applied to taking n derivatives of the $(x-a)^n$ term (we used both power rule and chain rule in class)

Geometric/graphical representations of the Taylor series

Maple can also output Taylor polynomials as visual representations. Here is the Taylor polynomial of degree 0 (e^5), which is always a horizontal line at the function value at the point, plotted along with the function $\left(e^x\right)$.

The sign tells us whether the function is above or below the x axis at a.

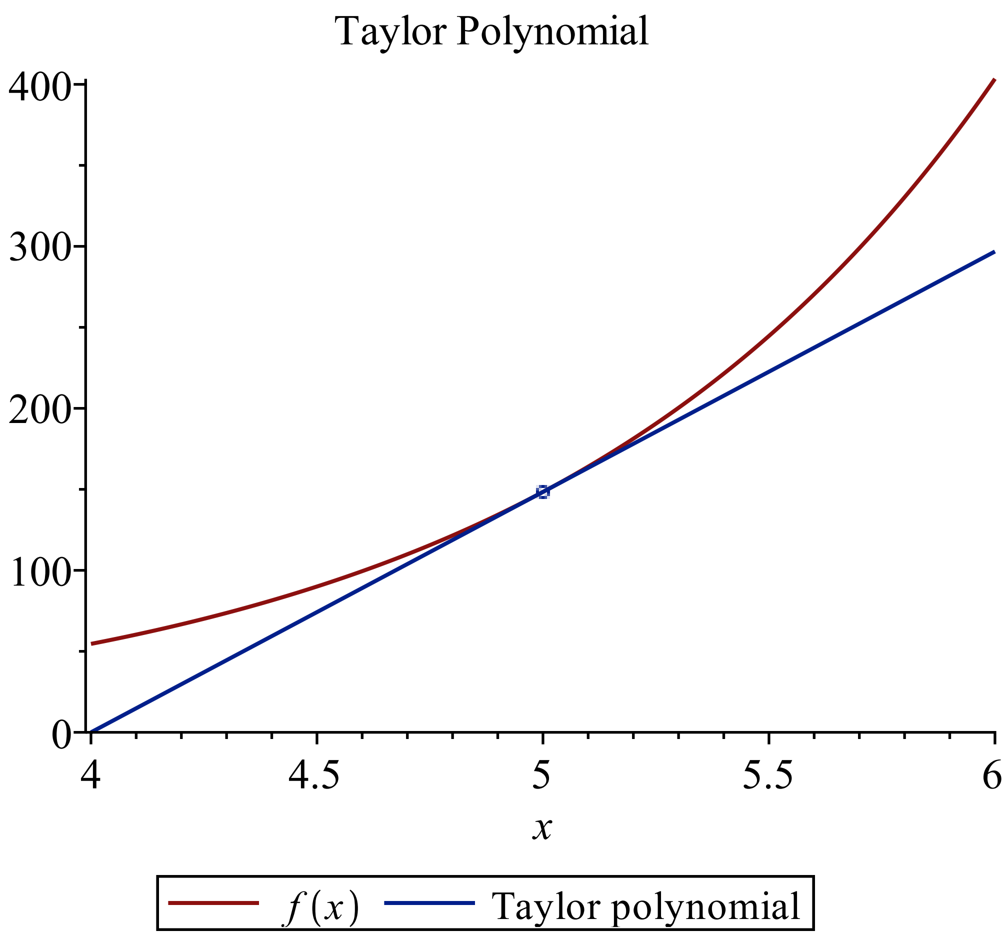
```
> TaylorApproximation(exp(x), x = 5, order=0,output=plot);
```



At $x = 5$, for the function $f(x) = e^x$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 0.

Here is the Taylor polynomial of degree 1 ($e^5 + e^5 \cdot (x - 5)$), which is the linear tangent line approximation, plotted along with the function (e^x). The sign of the Taylor term $\frac{f'(a)}{n!}(x - a) = e^5 \cdot (x - 5)$ tells us whether the function is increasing or decreasing at *a*.

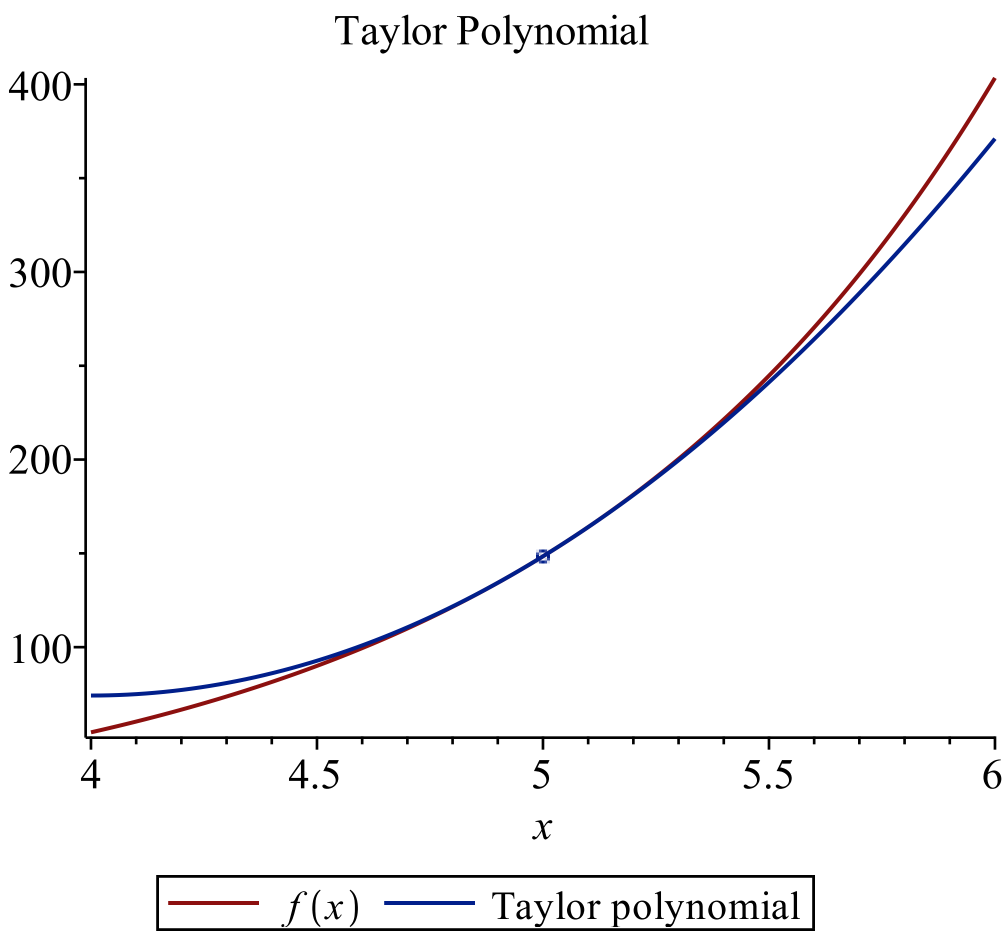
```
> TaylorApproximation(exp(x), x = 5, order=1,output=plot);
```



At $x = 5$, for the function $f(x) = e^{x}$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 1.

The sign of the next Taylor term (degree 2) tells us about concavity.

```
> TaylorApproximation(exp(x), x = 5, order=2,output=plot);
```



At $x = 5$, for the function $f(x) = \mathrm{e}^{x}$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 2.

<u>2) By hand activity</u>: Using the concavity of the graph above at x=0, will the sign of the degree 2 Taylor term be positive or negative? Answer in your notes.

positive since it is concave up

Taylor polynomials for sin(x) about x=0

<u>Notes activity</u> Here n=15, and we are computing the Taylor polynomial for sin(x) about x=0. Execute and then write the first few 3 terms in your notes, ie the Taylor polynomial of degree 5:

```
> TaylorApproximation(sin(x), x = 0, order=15);
```

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13} - \frac{1}{1307674368000}x^{15} \quad (5)$$

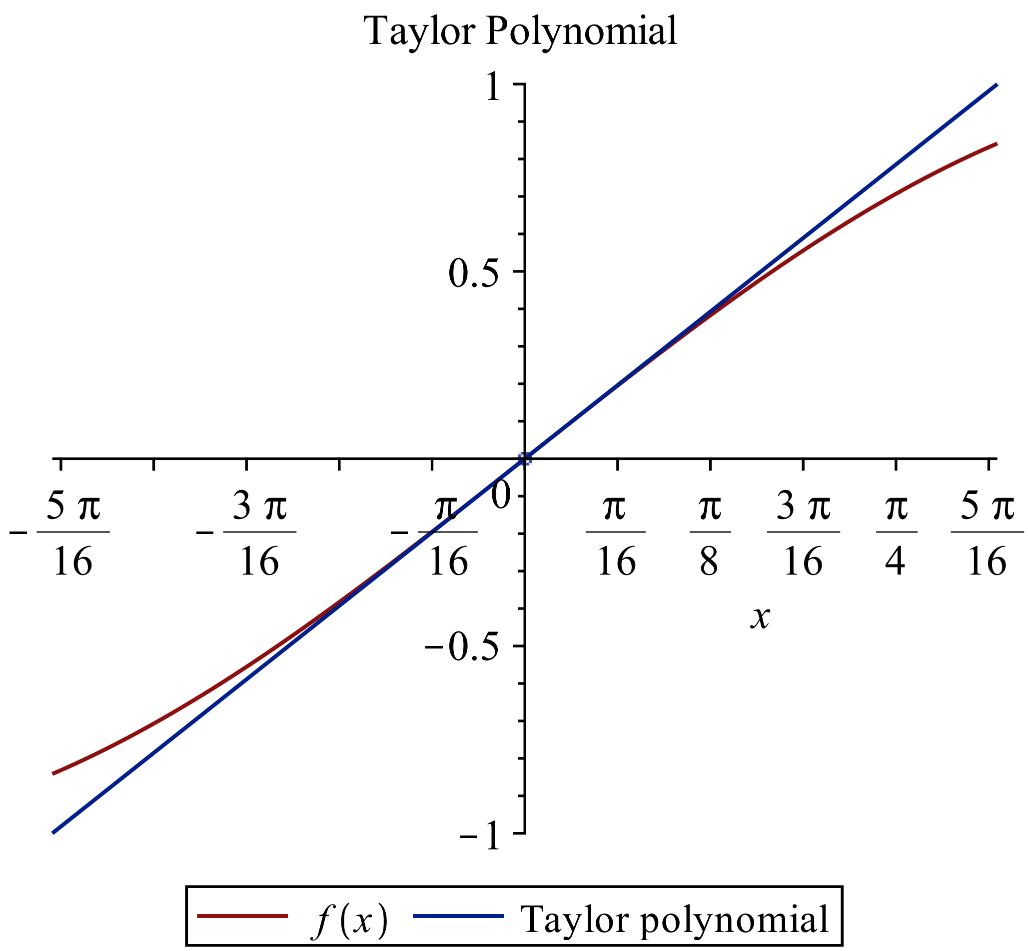
$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

Notice that instead of writing the Taylor series using factorial, like 3!, Maple multiplies out the factorials.

Also notice that there are no even terms here, so the degree 4 Taylor polynomial is the same as the degree 3 one--the Taylor term of degree 4 has a 0 coefficient.

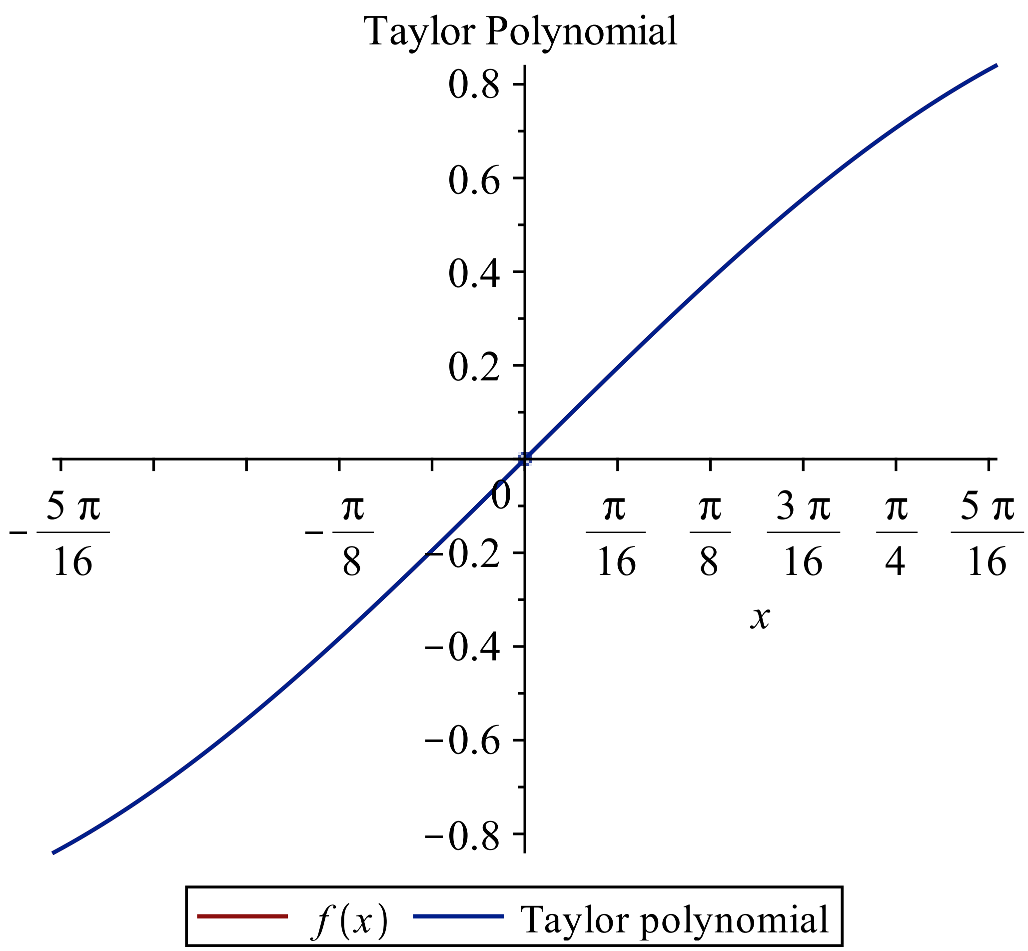
Notes activity First plot the linear approximation (Taylor polynomial of degree 1), and then plot of the 15th degree Taylor polynomial approximation, and then sketch them in your notes:

```
> TaylorApproximation(sin(x), x = 0, order=1,output=plot);
```



At $x = 0$, for the function $f(x) = \sin(x)$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 1.

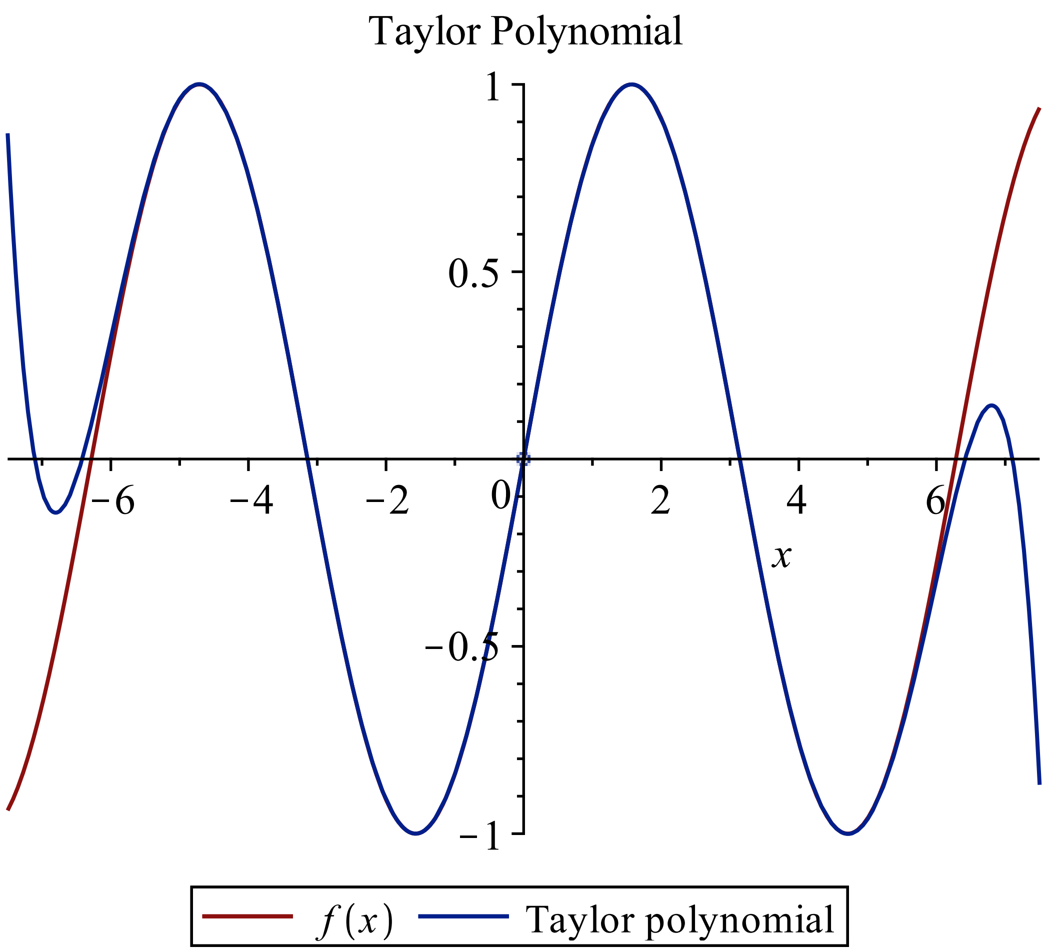
```
> TaylorApproximation(sin(x), x = 0, order=15,output=plot);
```



At $x = 0$, for the function $f(x) = \sin(x)$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 15.

Where did *f(x)* go here? Well it is such a good approximation at that point, that the difference between the polynomial and the function is not observable near *x*=0 within this small (local) plotting area. The functions would separate further away from 0:

```
> TaylorApproximation(sin(x), x = 0, -7.5..7.5, order=15,output=plot)
  ;
```



At $x = 0$, for the function $f(x) = \sin(x)$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 15.

Class webpage activity: To better process this local to global behavior, go back to the class highlights webpage and examine the Taylor series animation for sin(x)

Taylor polynomials for sin(x) about 1.

Want to change what x=a is? No problem. Here is the 2nd degree polynomial for sin(x) about x=1. Write the polynomial in your notes.

```
> TaylorApproximation(sin(x), x = 1, order=2);
```

$$\frac{\sin(1)}{2} + \cos(1)\,x - \cos(1) - \frac{\sin(1)\,x^2}{2} + \sin(1)\,x \qquad \textbf{(6)}$$

Where does the 1/2 coefficient of sin(1) come from? Well Maple collects like terms.

3) By hand activity: Write the table in your notes and fill in three rows, for n=0, n=1, and n= 2.

$f^n(x)$ stands for the nth derivative of $f(x)$ = sin(x), left as a function of x, while a=1 is plugged into $f^n(a)$.

n	$f^n(x)$	$f^n(a)$	Taylor term $\frac{f^n(a)}{n!}(x-a)^n$
0	sin(x)	sin(1)	$\frac{\sin(1)}{0!}(x-1)^0 = \sin(1)$
1	cos(x)	cos(1)	$\frac{\cos(1)}{1!}(x-1)^1 = \cos(1)(x-1) = \cos(1)\,x - \cos(1)$
2	-sin(x)	-sin(1)	$\frac{-\sin(1)}{2!}(x-1)^2$

4) By hand activity: Compute the 2nd degree polynomial for sin(x) about x=1 by hand in your notes by adding up the Taylor terms.

$$\sin(1) + \cos(1)(x-1) + \frac{-\sin(1)}{2!}(x-1)^2$$

Note that if we foiled it out, and collected like terms, then we could show that these are the same, and I've done this in solutions:

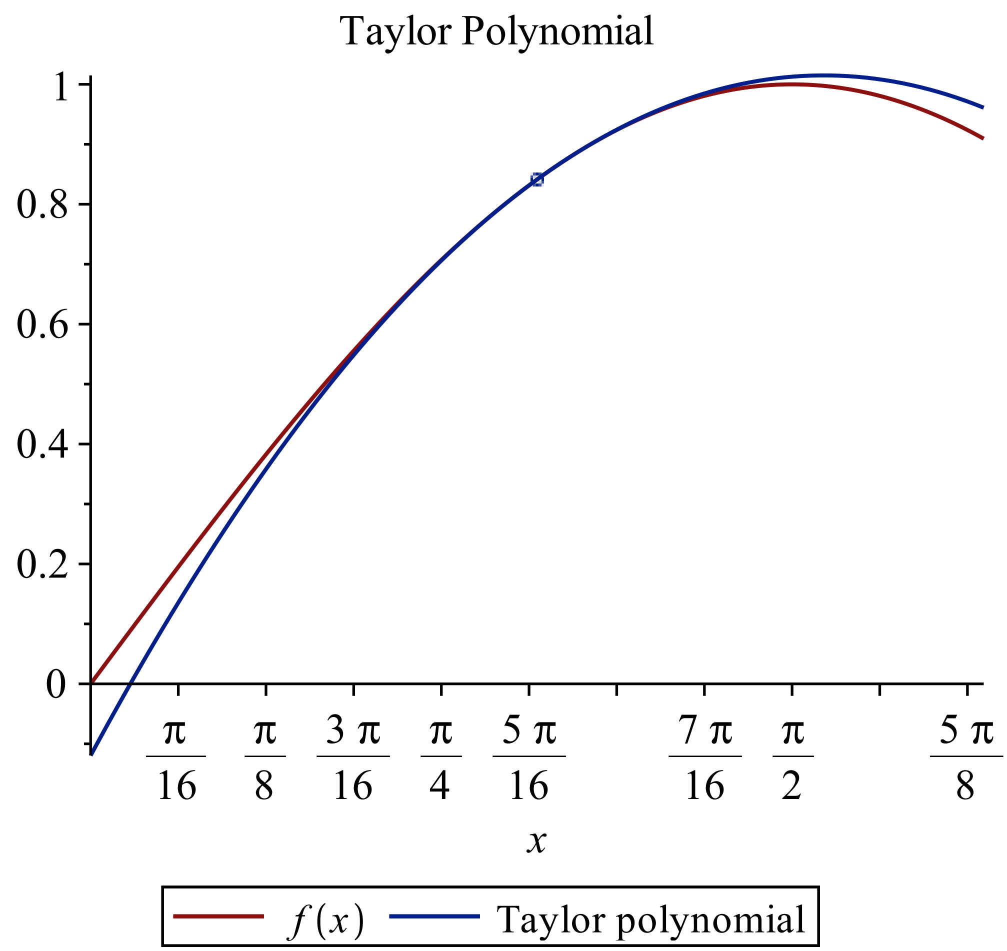
f(x) = sin(x) so f(1) = sin(1) and the first Taylor term is sin(1)
f'(x) = cos(x) so f'(1) = cos(1) and the second Taylor term is cos(1) (x-1)
$f^{(2)}$(x) = -sin(x) so $f^{(2)}$(1) = -sin(1) and the 3rd Taylor term is -sin(1)$\frac{1}{2!}\ (x-1)^2$
So the 2nd degree Taylor polynomial for sin(x) about x=1 is

sin(1) + cos(1) (x-1) + $-\sin(1)\frac{1}{2!}\ (x-1)^2$
Maple looks a bit different. To see they are the same, let's multiply out what we obtained by hand:
sin(1) + cos(1) (x-1) + $-\sin(1)\frac{1}{2!}\ (x-1)^2$
= sin(1) + cos(1) (x) -cos(1) + $-\frac{\sin(1)}{2}\ (x^2 - 2x + 1)$
= sin(1) + cos(1) (x) -cos(1) $-\frac{\sin(1)}{2}\ x^2 + \sin(1)\ x - \frac{\sin(1)}{2}$
Now combine the first and last term to see that

$$= \frac{1}{2}\sin(1) + \cos(1)\,x - \cos(1) - \frac{1}{2}\sin(1)\,x^2 + \sin(1)\,x$$
which matches Maple.

Notes activity: Let's compare with the plot. Execute and then sketch the degree 2 Taylor polynomial in your notes.

```
> TaylorApproximation(sin(x), x = 1, order=2,output=plot);
```

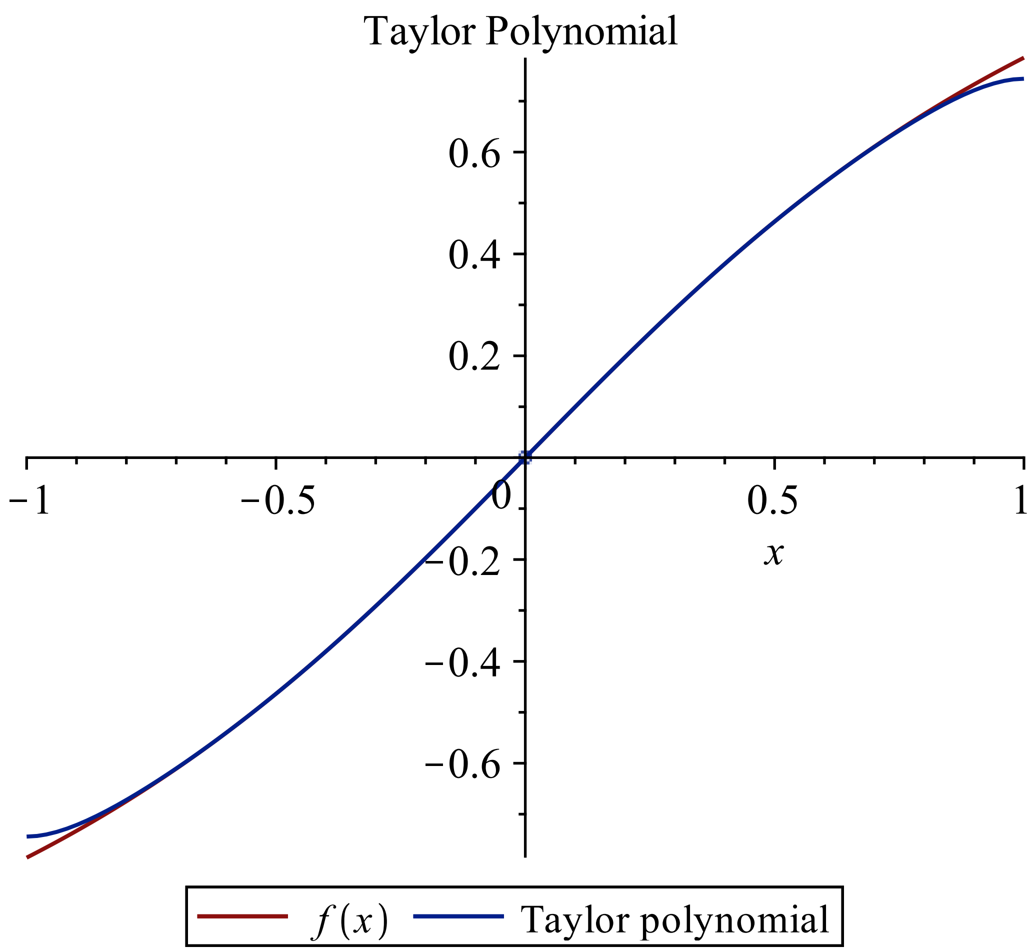


At $x = 1$, for the function $f(x) = \sin(x)$, a graph of $f(x)$
and the approximating Taylor polynomial(s) of degree(s) 2.

Taylor polynomials for arctan(*x*) about *x*=0

<u>Notes activity</u>: Sketch this plot in your notes.

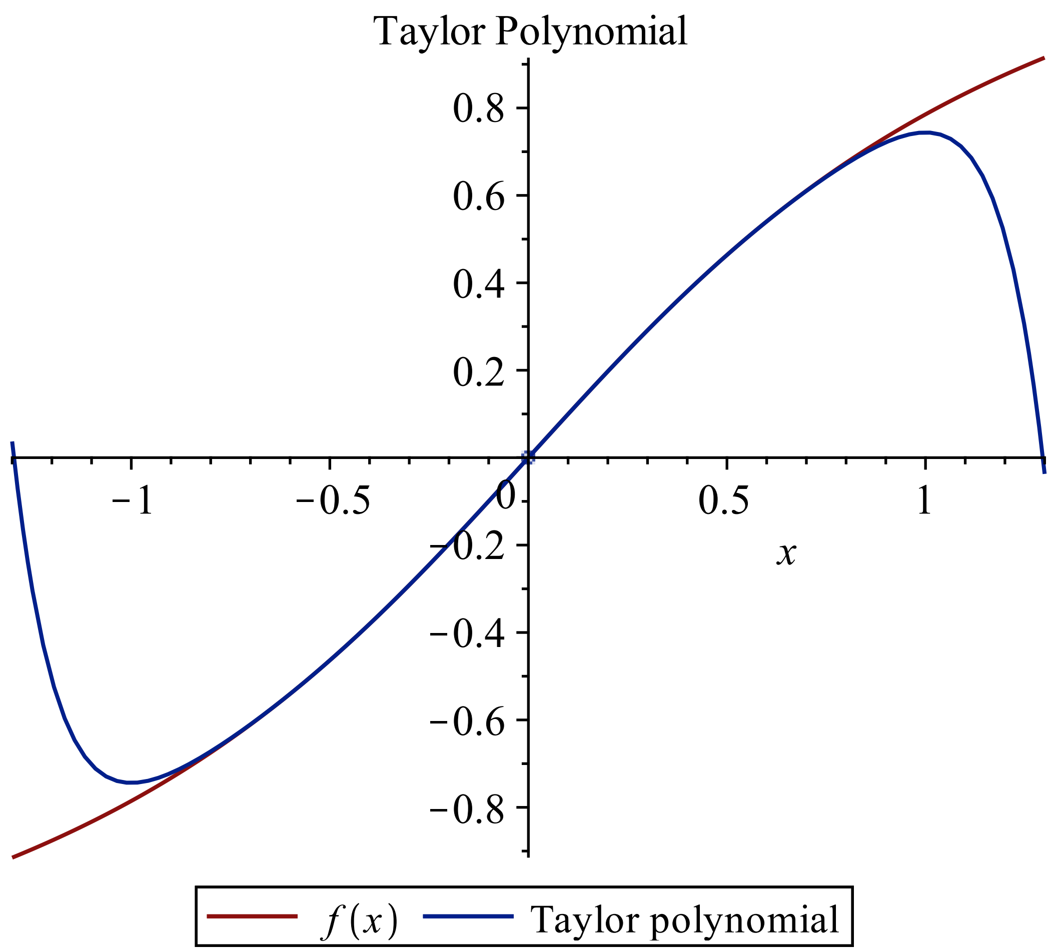
```
> TaylorApproximation(arctan(x), x = 0, order=11,output=plot);
```



At $x = 0$, for the function $f(x) = \arctan(x)$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 11.

The degree 11 Taylor polynomial approximates the function very well between -1 and 1. However, outside of that range it doesn't approximate the function well:

```
> TaylorApproximation(arctan(x), x = 0, -1.3..1.3, order=11,output=
  plot);
```



At $x = 0$, for the function $f(x) = \arctan(x)$, a graph of $f(x)$ and the approximating Taylor polynomial(s) of degree(s) 11.

5) By hand activity: Using the graph above, approximate the error between the Taylor polynomial of degree 11 and the function on the far right side of the graph (x=1.3). To do this, estimate the y-value difference from the tick marks on the graph, and write your estimate in your notes. We'll be computing other errors graphically like we did here, as well as algebraically in 10.4.

We look at x=1.3 and measure the y-value difference, which is approximately .92 or similar (you might have said .9 for example), since the Taylor polynomial goes slightly below the axis and the function value goes up past the .8 tickmark, up to the .9 tickmark.

Maple activity: Modify the Maple command from above to find the 11th degree polynomial that approximates arctan(x) at x=0.
So you'll need to change the function, the x value, and the order in the command. So modify sin(x), 1, and 2 below:

```
> TaylorApproximation(arctan(x), x = 0, order=11);
```

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} \qquad (7)$$

6) By hand activity: Next, using your Maple responses from above, write the general

summation form for the Taylor series in your notes, like James Gregory did for this series back in 1671. Start the index at 0.
Hint: n and $(2n+1)$ will both be useful.

Notice that the index of the series and the degree of the Taylor polynomial can be different, even as people may use n to stand for two different things!

We can see that $(-1)^n$ will work for the alternating signs of the series, because the first term of the series (starting at 0), x, has a positive sign, while the second term of the series (for an index of 1) has a negative sign.

Since the first term is x for the index starting at 0, and we want odd powers, $2n+1$ will work for the power and the denominator of the fraction:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

Using the ratio test on this series representation, we can compute the radius of convergence, just like we did in 9.5 on power series, since a Taylor series is a power series:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{1}{2\cdot(n+1)+1}x^{2\cdot(n+1)+1}}{(-1)^n\frac{1}{2n+1}x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{2n+2+1}x^{2n+2+1}}{\frac{1}{2n+1}x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{2n+3}x^{2n+3}}{\frac{1}{2n+1}x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}x^{2n+3-(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}x^{2n+3-(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}x^{2n+3-2n-1}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}x^2\right| = \lim_{n\to\infty}\frac{2n+1}{2n+3}x^2$$

Since this approaches infinity over infinity, we use L'Hopitals to continue, by taking the derivative of the numerator divided by the derivative of the denominator, with respect to n

$$= \lim_{n\to\infty}\frac{2}{2}x^2 = x^2$$

The ratio test converges when $L < 1$. Here this means that $L = x^2 < 1$.
So $-1 < x^2 < 1$. This tells us that $-1 < x < 1$. Thus the radius of convergence $r = 1$.

Ask me any questions on the above steps and show me your paper responses as you continue to:

<u>Class webpage activity:</u> Go back to the class highlights webpage and examine the Taylor series animation for arctan(x) to better appreciate the radius of convergence.