Differential Equations and Slope Fields in Maple

Goal: Explore differential equations using a variety of representations (numerical, symbolic, and graphical) through pattern exploration assisted by appropriate technology, including the computer algebra system Maple, which is one of the course goals.

In 11.1 we check whether a given function is a solution of a differential equation, an equation involving derivatives or differentials, (or whether intitial conditions work) by taking derivatives and testing to check if we satisfy the differential equation.

In 11.2 we use visualizations of slopes in order to create our own solutions. A slope field is a set of signposts directing us across the plane. For example, a slope of 1 at a point has a tangent line that is 45 degrees from the horizontal and is directed upward to the right because of the positive slope. We can plug the coordinates of a point (x,y) or (t,y) into a differential equation to obtain a numerical value and sketch a direction from there.

1. By-hand checking a solution (11.1)

Check that e^{t+3} is a solution of the differential equation $\frac{dy}{dt} = y$ by taking the derivative and checking whether it satisfies the equation. Show work in your notes.

In fact, any constant $e^{t+constant}$ is also a solution.

2. By-hand checking an initial condition (11.1)

Can $y = e^{t+3}$ pass through the point $(t,y) = (0, -1)$?

We plug in for t and y to see whether $-1 = e^{0+3}$ is possible. Write your response in your notes.

3. Graphically generating a slope by-hand (11.2)

While the above solution can't pass through the point (0,-1), there are other solutions to this DE that can pass through (0,-1). We'll graphically generate the start of what that solution would look like very very close to that point by creating a graphical tangent line.

First plug in to the right hand side of the differential equation (y in this case) to compute the numerical slope at the point $(t,y) = (0,-1)$. Then sketch by-hand the tangent line.

Note that as we talked about with Taylor series, if we are very very close to the center, a Taylor polynomial (like the linear approximation) looks like the function. We are only using one derivative here, so the Lagrange Error estimate/Taylor's Theorem would show us that we have to be quite close. We investigate more on this issue in 11.3 via numerical approximations in Euler's method.

The remaining worksheet combines aspects from 11.1 and 11.2:

4. Maple's slope diagram and solution for $\frac{dy}{dt} = y$

Maple can create slope field diagrams very quickly using the DEplot package, which relies on a variable t in place of x.

Notice that in the code below, diff(y(t),t)=y(t) is used to represent the differential equation $\frac{dy}{dt} = y$.

```
> with(DEtools):
  DEplot(diff(y(t),t) = y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =
  .5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```

Compare the what you drew in #3 with what Maple drew near y = -1. They should look similar.

Each slope vector represents the direction and magnitude:
horizontal line has slope 0.
vertical line has infinite slope.
45◦ positive slope up to right has a slope of 1.
-45◦ negative slope down to right has a slope of -1, like in #3.

Other slopes fall somewhere in between these, i.e. a slope of 2 is a bit steeper than the 45◦ angle and still slopes up the right right.

The solution to the differential equation is called an **<u>equilibrium</u>** solution if the derivative is zero everywhere, represented as a horizontal line. We see that y=0 is an equilibrium solution here. It is classified as **<u>unstable</u>**, because a small perturbation off of it results in solutions that move away from it. On the other hand, **<u>stable</u>** equilibrium solutions would have solutions nearby tending towards them as we'll see below. Sometimes equilibrium solutions are <u>neither</u> stable nor unstable.

Maple can also solve differential equations for algebraic solutions. Execute the following:

```
> dsolve(diff(y(t),t) = y(t));
```

Maple gives us _C1, a generic constant. While this looks a bit different than e^{t+3} the solutions are algebraically equivalent: $e^{t + constant} = e^t e^{constant}$ (see the powers of x of the Algebra, Geometry & Trigonometry portion of the review sheet). Now $e^{constant}$ is also a constant. So in this case Maple's _C1 represents our $e^{constant}$.

Ask me any questions you have on the termonology or algebraic equivalences.

5. Interpreting Maple's slope diagram for Equilibrium Solutions of $\frac{dy}{dt} = -y$

In the command below I modified the differential equation to add one negative to the equal portion of the DE (= -y(t) replaces =y(t)). The other instance of y(t) in the command remains the same. Execute:

```
> DEplot(diff(y(t),t) = -y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =
  .5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```

Notice that once we added the negative sign in the above command the DEplot changed.
Are there any equilibrium solutions?
If so, are they stable, unstable, or neither?

6. Fixing a Problem with a Singularity in $\frac{dy}{dt} = \frac{y}{t}$ at t=0.

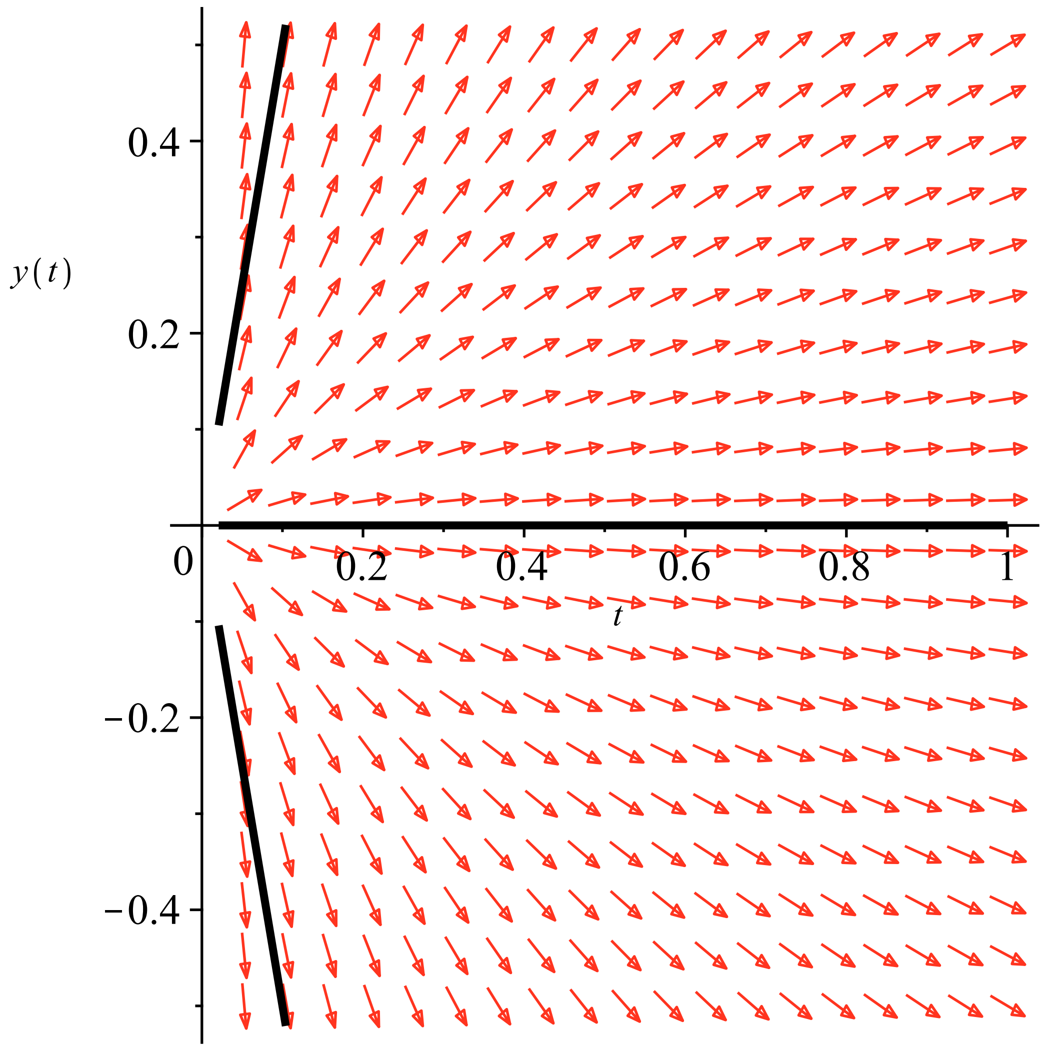
Execute the differential equation in the command below. Notice that I replaced = y(t) with =y(t)/t.

```
> DEplot(diff(y(t),t) = y(t)/t, y, t = -1 .. 1, y = -1 .. 1, [y(0) =
  .5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```

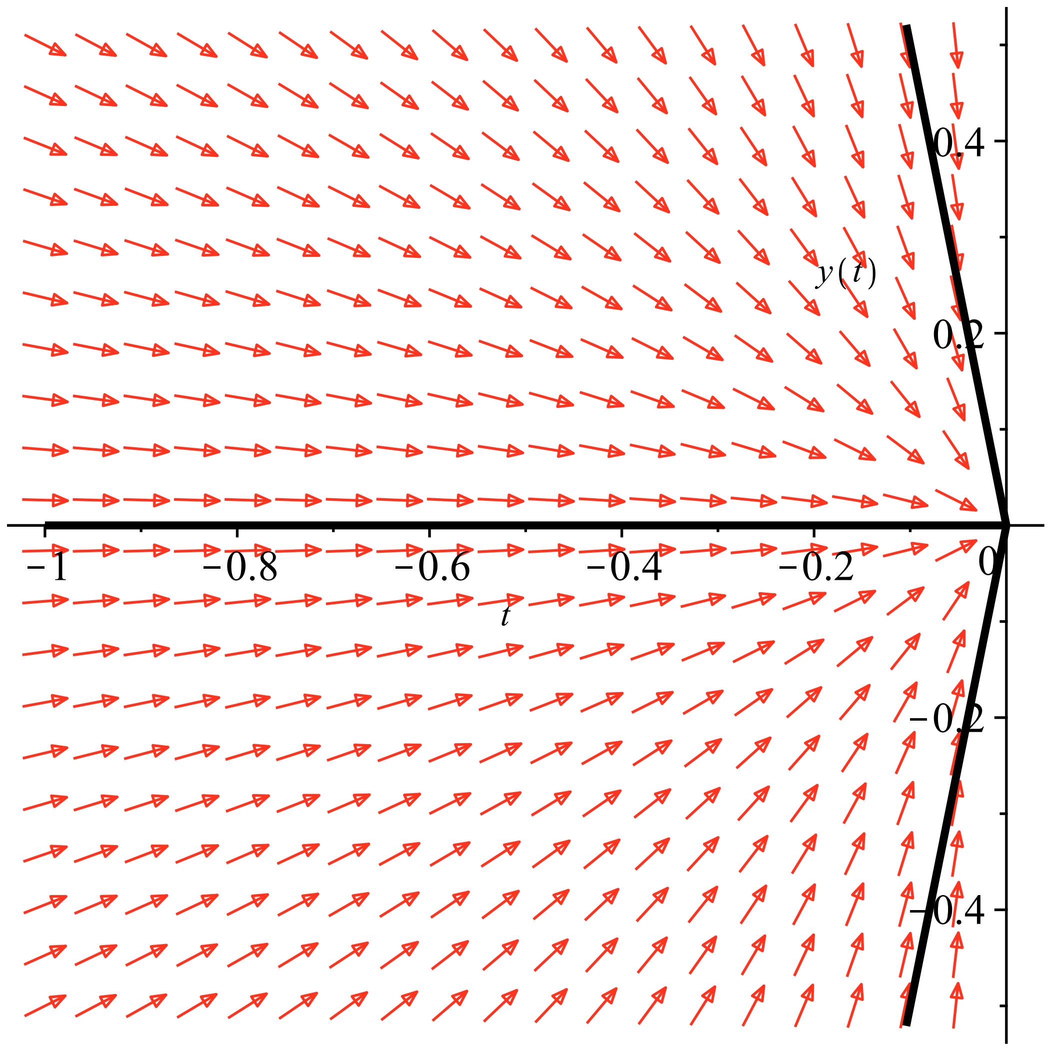
Activity: We receive an error message because we have undefined slopes when t (or x) is 0, and the initial conditions all specify y(0).
Create a small perturbation of the problematic initial conditions by replacing all three instances of y(0) inside the square brackets above with y(.1) in order to remove the error message and obtain the DEplot.

Notice that there is an equilibrium solution for $\frac{dy}{dx} = \frac{y}{t}$ when y=0 as we see the constant horizontal line there (t=0 was the problem above, not y=0 which is defined everywhere except when t=0). To the right of the y-axis, solutions that start nearby y=0 move away from it, but they do the opposite to the left of the y-axis as they move towards the horizontal line. So this neither a stable nor unstable solution.

Solutions move away from the equilibrium in the first picture, and towards it in the second:



0.4
0.2
y(t)
0
0.2
0.4
0.6
0.8
1
t
−0.2
−0.4



7. Modifying Maple's DEplot command for $\frac{\mathbf{d}y}{\mathbf{d}t} = ty$.

<u>Activity</u>: Modify the differential equation in the command below. **<u>Replace</u>** = y(t) with =t*y(t)

```
> DEplot(diff(y(t),t) = y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =
  .5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```

8. Interpreting Maple's slope diagram for Equilibrium Solutions of $\frac{\mathbf{d}y}{\mathbf{d}t} = ty$.

Are there any equilibrium solutions? Is it stable (nearby solutions always tend towards it), unstable (nearby solutions always tend away from it), or neither (a combination). Hint: check to the right of the y-axis for the behavoir or those nearby solutions, and then to the left of the y-axis.

9. $\frac{dy}{dt}$ = your creation!

We can use all sorts of functions although they must be expressed like Maple does, like
=(y(t))^2
=sec(y(t))
=t*sec(y(t))
=exp(t)*sec(y(t))
for instance

Activity: Modify the differential equation in the command below. Replace = y(t) with a differential equation of your creation using *t* and/or *y(t)*. It could be one of the ones just above, or something else completely.

```
> DEplot(diff(y(t),t) = y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =
  .5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```

If Maple gives you an error then you might need to modify the initial conditions or domain or range if your function has undefined values.

List the DE in your notes.

10. Go back to the class highlights webpage for the CoursEval link.