Evelyn Boyd Granville

second Black woman we know of—PhD in mathematics

Image 1 Credit: `http://www.visionaryproject.org/granvilleevelyn/`

Image 2 Credit: Marge Murray. Courtesy of Evelyn Boyd Granville

...this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space

Rabbits and chickens have been placed in a cage. You count 48 feet and seventeen heads. Let x = rabbits, y= chickens. heads? feet? solve—three different methods?

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row [like $r_2' = -3r_1 + r_2$].
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Gaussian Elimination—Roughly

1. Use x_1 in the first equation to eliminate the other x_1 terms below it
2. Ignore eq 1 and use x_2 in the 2nd equation to eliminate the terms below it
3. . . . triangular (interchange as needed)

EBG Using Gaussian Elimination (Echelon Form)

Rabbits and chickens have been placed in a cage. You count 48 feet and seventeen heads. Let x = rabbits, y= chickens.

$$x + y = 17$$
$$4x + 2y = 48$$

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  (7, 10) is the *unique* solution (point in $\mathbb{R}^2$)

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$$\begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix}$$ **augmented matrix**

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$$\begin{bmatrix} 1 & 1 & 17 \\ 4 & 2 & 48 \end{bmatrix} \xrightarrow{r_2'=-4r_1+r_2} \begin{bmatrix} 1 & 1 & 17 \\ -4\cdot 1 + 4 & -4\cdot 1 + 2 & -4\cdot 17 + 48 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 17 \\ 0 & -2 & -20 \end{bmatrix}$$

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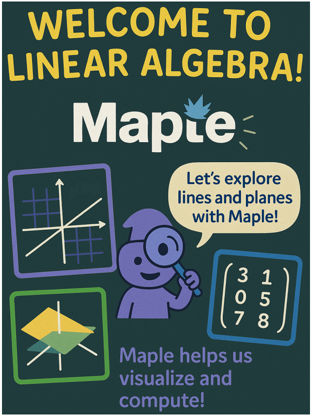
Evelyn Boyd Granville in Maple

```
with(LinearAlgebra): with(plots):
implicitplot(x+y=17, 4*x+2*y=48,x=-10..10, y = 0..40);
EBG:=Matrix([[1,1,17],[4,2,48]]);
GaussianElimination(EBG);
ReducedRowEchelonForm(EBG);
```



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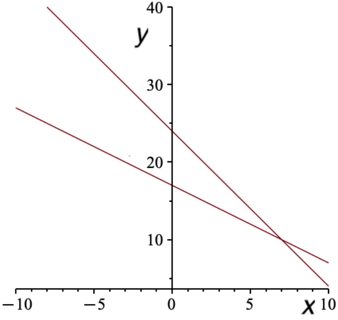
Compare and Contrast Representations



https://www.etsy.com/listing/506994463/math-pun-coffee-mug-funny-joke-mug-funny

- implicitplot graphs the rows as lines to reveal their concurrent intersection at $(x, y) = (7, 10)$
- rref provides the numerical coordinates via a reduced augmented matrix while backsub is needed from Gaussian
- all three methods reveal one unique solution
- with children, cutouts to match up the heads to the feet

h as Coefficients of Variables

$x + hy = 0$

$hx + y = 0$

Use Gaussian elimination $-h$ equation 1 + equation 2

Use x term in eq1 to eliminate terms below it via $r_h' = cr_1 + r_k$

Augmented matrix:

$$\begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \end{bmatrix} \xrightarrow{r_2' = -hr_1 + r_2} \begin{bmatrix} 1 & h & 0 \\ ? & ? & ? \end{bmatrix} =$$

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$$\begin{bmatrix} 1 & h & 0 \\ -h \cdot 1 + h & -h \cdot h + 1 & -h \cdot 0 + 0 \end{bmatrix} = \begin{bmatrix} 1 & h & 0 \\ 0 & -h^2 + 1 & 0 \end{bmatrix}$$

How many solutions?

Gaussian elimination

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How many solutions? Gaussian elimination

- ~~($0x + 0y$=nonzero) **inconsistent**, 0 concurrent solutions~~
- row 2: $(-h^2 + 1)y = 0$
  $h = \pm 1$ $0 = 0$ no info. $h = 1$ $x + y = 0$ repeated. ∞ sols
  $h = -1$ $x - y = 0$ and $-x + y = 0$ same line. ∞ sols
  $h \neq \pm 1, y = \frac{0}{-h^2+1} = 0$ and row 1: $x + h(0) = 0$ so (0,0)

h as Coefficients of Variables

$x+hy=0$

$hx+y=0$

$$\begin{bmatrix} 1 & h & 0 \\ h & 1 & 0 \end{bmatrix} \xrightarrow{r_2'=-hr_1+r_2} \begin{bmatrix} 1 & h & 0 \\ 0 & -h^2+1 & 0 \end{bmatrix}$$

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in Maple?

`heqs:=Matrix([[1,h,0],[h,1,0]]);`

`GaussianElimination(heqs);`

~~`ReducedRowEchelonForm(heqs);`~~

Systems of Equations with 2 Unknowns

What are the possible solutions of a linear system with two equations and two unknowns? Can you provide examples? More generally, why do these represent all the possibilities?

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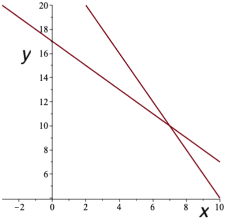

$$x + y = 17$$
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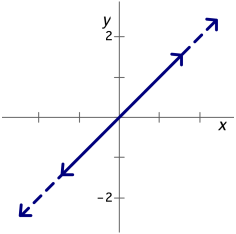
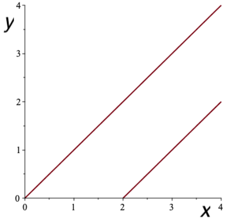


$$x - y = 0$$
$$-x + y = 0$$



$$x - y = 0$$
$$x - y = 2$$

Systems of Equations with 3 Unknowns

2 equations 3 unknowns: If possible, draw a picture of a linear system with two equations and three unknowns that has a unique solution. If this is not possible, explain why not.

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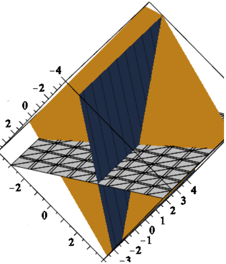

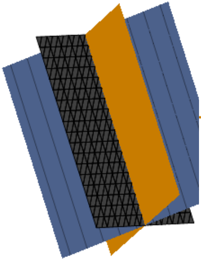

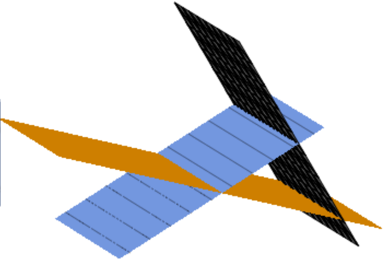

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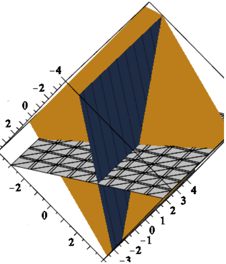
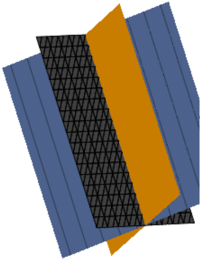
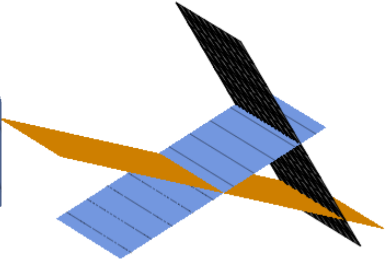


1 solution
corner of room



infinite solutions
book spine



0 solutions
hands + table

$$\begin{aligned} x+2y+3z&=9\\ 2x-y+z&=8\\ 3x-z&=3 \end{aligned} \qquad \begin{array}{cccc} x & y & z & = \\ \hline \end{array}$$

$$\begin{array}{cccc} x & y & z & = \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow{r_2'=-2r_1+r_2}$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 9 \\ -2\cdot 1+2=0 & -2\cdot 2-1=-5 & -2\cdot 3+1=-5 & -2\cdot 9+8=-10 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

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$$\xrightarrow{r_3'=-3r_1+r_3}$$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ -3\cdot 1 + 3 = 0 & -3\cdot 2 + 0 = -6 & -3\cdot 3 - 1 = -10 & -3\cdot 9 + 3 = -24 \end{array}\right]$$

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$$\xrightarrow{r_3' = -\frac{6}{5} r_2 + r_3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ -\frac{6}{5} \cdot 0 + 0 & -\frac{6}{5} \cdot -5 - 6 & -\frac{6}{5} \cdot -5 - 10 & -\frac{6}{5} \cdot -10 - 24 \end{bmatrix}$$

The original system has the same solution set as the reduced

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x - z &= 3 \end{aligned} \qquad \begin{array}{cccc} x & y & z & = \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ -5 \\ 0 \end{matrix} & \begin{matrix} 3 \\ -5 \\ -4 \end{matrix} & \left.\begin{matrix} 9 \\ -10 \\ -12 \end{matrix}\right] \end{array}$$

row 3:$-4z = -12$ so $z = 3$

row 2: $-5y - 5z = -10$ so $-5y - 5 \cdot 3 = -10$. Then $-5y = 5$ and so $y = -1$.

Substituting $z = 3$ and $y = -1$ into equation 1 gives

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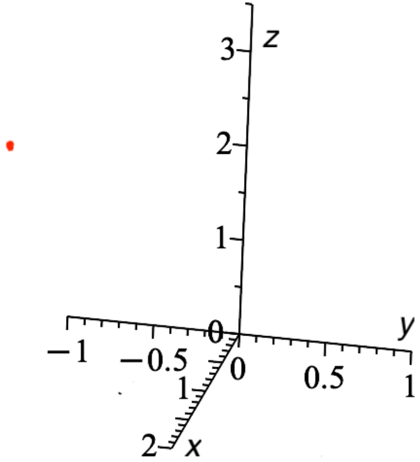
$$\begin{aligned} x + 2 \cdot -1 + 3 \cdot 3 &= 9 \\ x &= -2 \cdot -1 - 3 \cdot 3 + 9 \\ x &= 2 \end{aligned}$$

Therefore the unique solution to our linear system is $(x, y, z) = (2, -1, 3)$.

Thinking about the Geometry in 3-space

$(x, y, z) = (2, -1, 3)$

x out of the board, y horizontal to the right, z up



Thinking about the Geometry in 3-space

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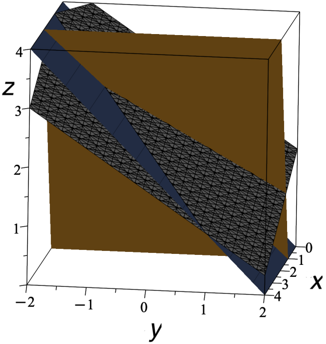
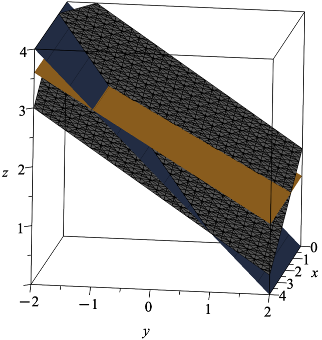
$$\begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 3x & & -z & = 3 \end{array} \qquad \begin{array}{llll} x & +2y & +3z & = 9 \\ 0 & -5y & -5z & = -10 \\ 0 & -6y & -10z & = -24 \end{array}$$





Thinking about the Geometry

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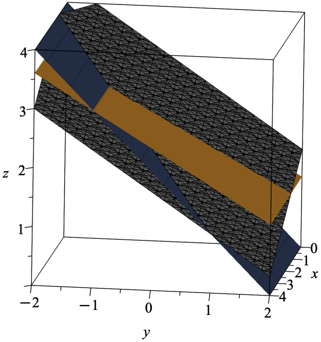
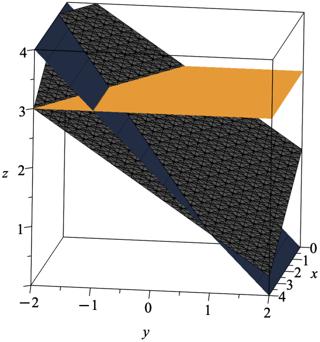
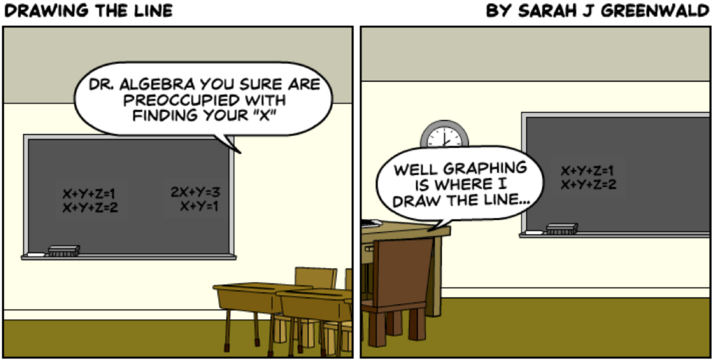




Systems of Equations with 3 Unknowns in Maple

```
implicitplot3d(x+2*y+3*z=9, 2*x-y+z=8, 3*x-z=3,x=-4..4, y =
-4..4, z=-4..4);
prob3planes:=Matrix([[1,2,3,9],[2,-1,1,8],[3,0,-1,3]]);
GaussianElimination(prob3planes);
ReducedRowEchelonForm(prob3planes);
```



Solutions?

How many solutions does a system have with the augmented matrix row-equivalent to

$$\begin{array}{cccc} x & y & z & = \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 3 \\ 1 \\ -1 \end{matrix} & \begin{matrix} 4 \\ 7 \\ -7 \end{matrix} & \left.\begin{matrix} 5 \\ -2 \\ 2 \end{matrix}\right] \end{array}$$

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Solutions?

How many solutions does a system have with the augmented matrix row-equivalent to

$$\begin{array}{c} \begin{array}{cccc} x & y & z & = \end{array} \\ \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & -1 & -7 & 2 \end{bmatrix} \end{array} \xrightarrow{r_3' = r_2 + r_3} \begin{array}{c} \begin{array}{cccc} x & y & z & = \end{array} \\ \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

If we changed the 2 in row 3 column 4 of the given augmented matrix to a 3, then how many solutions does the system have?

$$\begin{array}{c} \begin{array}{cccc} x & y & z & = \end{array} \\ \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & -1 & -7 & 3 \end{bmatrix} \end{array}$$

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$$\begin{array}{cccc} x & y & z & = \\ \hline 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & -1 & -7 & 2 \end{array} \xrightarrow{r_3'=r_2+r_3} \begin{array}{cccc} x & y & z & = \\ \hline 1 & 3 & 4 & 5 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 \end{array}$$

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