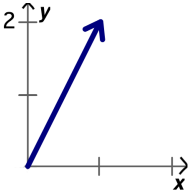
1.3 Vectors in $\mathbb{R}^n$

- ordered reals $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\\\vdots\end{bmatrix}$ rectangular coordinates

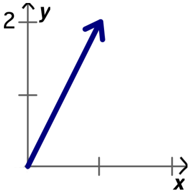
$$\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

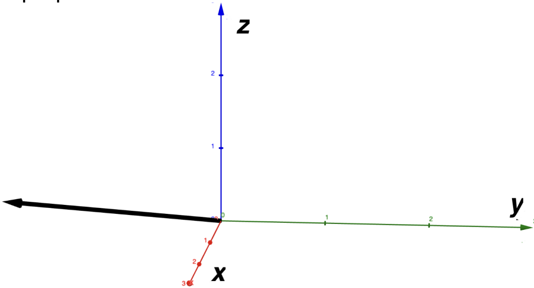


1.3 Vectors in $\mathbb{R}^n$

- ordered reals $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\\\vdots\end{bmatrix}$ rectangular coordinates

$$\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

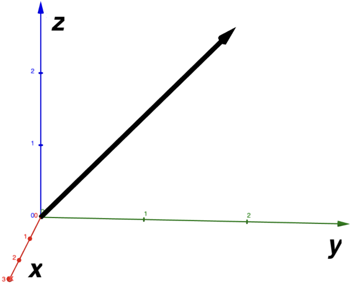




$$\vec{v}_2 = \begin{bmatrix}1\\-2\\.5\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

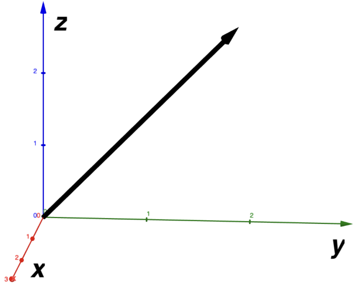
$$\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$



- scalar multiplication of a vector $c\vec{v} = c\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}c\\2c\end{bmatrix}$
- slope of the line a vector in $\mathbb{R}^2$ is on? $\dfrac{\Delta y}{\Delta x} = \dfrac{y-0}{x-0} = \dfrac{y}{x}$

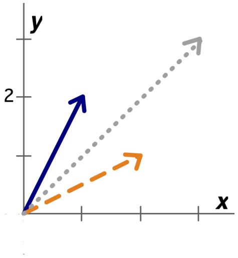
Vectors in $\mathbb{R}^2$ *and* $\mathbb{R}^3$

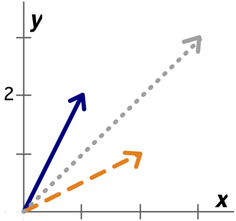
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



- scalar multiplication of a vector $c\vec{v} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix}$
- slope of the line a vector in $\mathbb{R}^2$ is on? $\dfrac{\Delta y}{\Delta x} = \dfrac{y - 0}{x - 0} = \dfrac{y}{x}$ $\dfrac{2c}{c}$
- algebra and geometry in $\mathbb{R}^3$ and $\mathbb{R}^n$

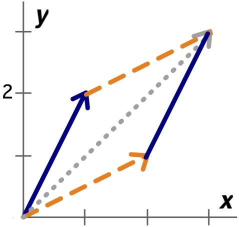
- addition of two vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

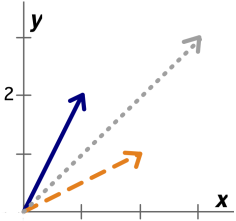




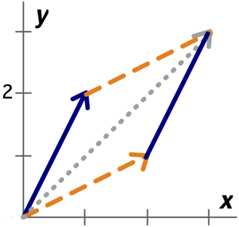
- addition of two vectors $\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}$

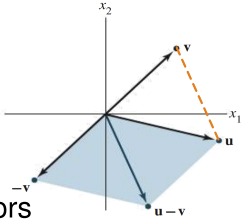
  diagonal of parallelogram if they are on different lines





- addition of two vectors $\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}$
  diagonal of parallelogram if they are on different lines





- subtraction of two vectors
  parallel to off diagonal of parallelogram
  if they are on different lines

Image 3: *Linear Algebra and Its Applications* by David Lay, Steven Lay and Judi McDonald

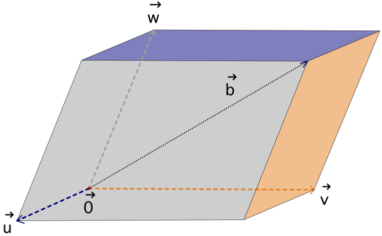
Linear Combinations: Addition & Scalar Mult

- $\vec{v}$ is a *linear combination* of $\vec{v}_1, ..., \vec{v}_n$ if $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, where the *weights* c_i are real.

Linear Combinations: Addition & Scalar Mult

- $\vec{v}$ is a *linear combination* of $\vec{v}_1, ..., \vec{v}_n$ if $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, where the *weights* c_i are real.

  Your weighted course average is a linear combination.



$$\vec{b} = \vec{u} + \vec{v} + \vec{w}$$

Is $\begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} .3 \\ .2 \\ .2 \\ .3 \end{bmatrix}, \begin{bmatrix} .4 \\ .3 \\ .2 \\ .1 \end{bmatrix} \right\}$?

A coffee shop offers two blends of coffees: House & Deluxe. Each is a blend of Brazil, Colombia, Kenya, & Sumatra roasts:

	House	Deluxe
Brazil	30%	40%
Columbia	20%	30%
Kenya	20%	20%
Sumatra	30%	10%

. Suppose we have 36 lbs of Brazil roast, 26 lbs of Columbia roast, 20 lbs of Kenya roast, and 18 lbs of Sumatra roast in stock and want to completely use it up. What represents the system?

a) $\text{lbs House}\begin{bmatrix} .3 \\ .2 \\ .2 \\ .3 \end{bmatrix} + \text{lbs Deluxe}\begin{bmatrix} .4 \\ .3 \\ .2 \\ .1 \end{bmatrix} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix}$

b) $\begin{bmatrix} .3\text{lbs House} + .4\text{lbs Deluxe} \\ .2\text{lbs House} + .3\text{lbs Deluxe} \\ .2\text{lbs House} + .2\text{lbs Deluxe} \\ .3\text{lbs House} + .1\text{lbs Deluxe} \end{bmatrix} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix}$

c) both

13intro.mw in Maple

$$\text{lbs House}\begin{bmatrix} .3 \\ .2 \\ .2 \\ .3 \end{bmatrix} + \text{lbs Deluxe}\begin{bmatrix} .4 \\ .3 \\ .2 \\ .1 \end{bmatrix} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix} \text{ vector eq.}$$

$$\begin{bmatrix} .3\text{lbs House} + .4\text{lbs Deluxe} \\ .2\text{lbs House} + .3\text{lbs Deluxe} \\ .2\text{lbs House} + .2\text{lbs Deluxe} \\ .3\text{lbs House} + .1\text{lbs Deluxe} \end{bmatrix} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix} \text{ row eqs. for system}$$

$$\begin{bmatrix} \frac{3}{10} & \frac{4}{10} & 36 \\ \frac{2}{10} & \frac{3}{10} & 26 \\ \frac{2}{10} & \frac{2}{10} & 20 \\ \frac{3}{10} & \frac{1}{10} & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 40 \\ 0 & 1 & 60 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ augmented matrix and reduction}$$

$$40\begin{bmatrix} \frac{3}{10} \\ \frac{2}{10} \\ \frac{2}{10} \\ \frac{3}{10} \end{bmatrix} + 60\begin{bmatrix} \frac{4}{10} \\ \frac{3}{10} \\ \frac{2}{10} \\ \frac{1}{10} \end{bmatrix} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix} \text{ vector eq. weights solved}$$

Linear Combinations and Span

- $\vec{v}$ is a *linear combination* of $\vec{v}_1, ..., \vec{v}_n$ if $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, where the *weights* c_i are real.

  A vector equation $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{b}$ has the same solution set as the linear system $[\vec{v}_1\,\vec{v}_2 \ldots \vec{v}_n|\vec{b}]$

Linear Combinations and Span

- $\vec{v}$ is a *linear combination* of $\vec{v}_1, ..., \vec{v}_n$ if $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, where the *weights* c_i are real.

  A vector equation $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{b}$ has the same solution set as the linear system $[\vec{v}_1\,\vec{v}_2 \ldots \vec{v}_n|\vec{b}]$

- The *span* of $\vec{v}_1, ..., \vec{v}_n$ is the set of all linear combinations, over all possible weights.

Linear Combinations and Span

- $\vec{v}$ is a *linear combination* of $\vec{v}_1, ..., \vec{v}_n$ if $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, where the *weights* c_i are real.

  A vector equation $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{b}$ has the same solution set as the linear system $[\vec{v}_1\,\vec{v}_2 \ldots \vec{v}_n|\vec{b}]$

- The *span* of $\vec{v}_1, ..., \vec{v}_n$ is the set of all linear combinations, over all possible weights.
  It is a linear space that we can find geometrically or algebraically using a generic vector or critical reasoning

  $$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \quad c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- 13intro.mw in Maple

$$\text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \text{ linear combinations } c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\begin{bmatrix}1&4&b_1\\2&5&b_2\\3&6&b_3\end{bmatrix}\xrightarrow[r_3'=-3r_1+r_3]{r_2'=-2r_1+r_2}\left[\begin{matrix}\boxed{1}&4&b_1\\-2\cdot 1+2&-2\cdot 4+5&-2\cdot b_1+b_2\end{matrix}\right.$$

$$\text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \text{ linear combinations } c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\begin{bmatrix}1&4&b_1\\2&5&b_2\\3&6&b_3\end{bmatrix}\xrightarrow[r_3'=-3r_1+r_3]{r_2'=-2r_1+r_2}\begin{bmatrix}\boxed{1}&4&b_1\\-2\cdot 1+2&-2\cdot 4+5&-2\cdot b_1+b_2\\-3\cdot 1+3&-3\cdot 4+6&-3\cdot b_1+b_3\end{bmatrix}=$$

$$\begin{bmatrix}\boxed{1}&4&b_1\\0&-3&-2b_1+b_2\\0&-6&-3b_1+b_3\end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \text{ linear combinations } c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\begin{bmatrix}1 & 4 & b_1\\2 & 5 & b_2\\3 & 6 & b_3\end{bmatrix}\xrightarrow[r_3'=-3r_1+r_3]{r_2'=-2r_1+r_2}\begin{bmatrix}\boxed{1} & 4 & b_1\\-2\cdot1+2 & -2\cdot4+5 & -2\cdot b_1+b_2\\-3\cdot1+3 & -3\cdot4+6 & -3\cdot b_1+b_3\end{bmatrix}=$$

$$\begin{bmatrix}\boxed{1} & 4 & b_1\\0 & -3 & -2b_1+b_2\\0 & -6 & -3b_1+b_3\end{bmatrix}\xrightarrow{r_3'=-2r_2+r_3}$$

$$\begin{bmatrix}\boxed{1} & 4 & b_1\\0 & \boxed{-3} & -2b1+b2\\0 & 0 & -2(-2b_1+b_2)-3b_1+b_3\end{bmatrix}=$$

$$\begin{bmatrix}\boxed{1} & 4 & b_1\\0 & \boxed{-3} & -2b1+b2\\0 & 0 & b_1-2b_2+b_3\end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \text{ linear combinations } c_1\begin{bmatrix}1\\2\\3\end{bmatrix}+c_2\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\begin{bmatrix}1&4&b_1\\2&5&b_2\\3&6&b_3\end{bmatrix}\xrightarrow[r_3'=-3r_1+r_3]{r_2'=-2r_1+r_2}\begin{bmatrix}\boxed{1}&4&b_1\\-2\cdot1+2&-2\cdot4+5&-2\cdot b_1+b_2\\-3\cdot1+3&-3\cdot4+6&-3\cdot b_1+b_3\end{bmatrix}=$$

$$\begin{bmatrix}\boxed{1}&4&b_1\\0&-3&-2b_1+b_2\\0&-6&-3b_1+b_3\end{bmatrix}\xrightarrow{r_3'=-2r_2+r_3}$$

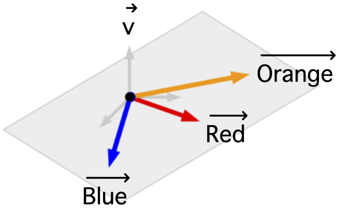
$$\begin{bmatrix}\boxed{1}&4&b_1\\0&\boxed{-3}&-2b1+b2\\0&0&-2(-2b_1+b_2)-3b_1+b_3\end{bmatrix}=$$

$$\begin{bmatrix}\boxed{1}&4&b_1\\0&\boxed{-3}&-2b1+b2\\0&0&b_1-2b_2+b_3\end{bmatrix} \quad b_1-2b_2+b_3=0 \text{ consistent}$$

$7-2(8)+9=0$ so $\begin{bmatrix}7\\8\\9\end{bmatrix}$ is in the span, a plane in $\mathbb{R}^3$

What's the Span?

$\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ has heard about the exclusive club $xy0$-plane. One night he stands outside trying to get in.



$\vec{v}$: Can I join in on your span?

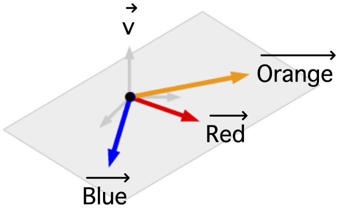
$\overrightarrow{\text{Red}}$: I'm sorry, but you'll never be in our span...

$\vec{v}$: Can't we all just get along (linearly)?

$\overrightarrow{\text{Orange}}$: All is not lost. Collectively we can open a new club!

What's the Span?

$\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ has heard about the exclusive club $xy0$-plane. One night he stands outside trying to get in.



$\vec{v}$: Can I join in on your span?

$\vec{\text{Red}}$: I'm sorry, but you'll never be in our span...

$\vec{v}$: Can't we all just get along (linearly)?

$\vec{\text{Orange}}$: All is not lost. Collectively we can open a new club!

$\vec{\text{Red}}$: We have been looking to expand.

$\vec{\text{Blue}}$: It'll be open access! (in $\mathbb{R}^3$)

Image adapted from June Lester
http://thejuniverse.org/PUBLIC/LinearAlgebra/LOLA/spans/three.html

Collections of Vectors in $\mathbb{R}^2$

What do the collection of column vectors $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, for c_1 and c_2 real, have in common?

a) They are linear combinations of the vectors of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, of the form $\begin{bmatrix} c_1 + 2c_2 \\ c_1 + 2c_2 \end{bmatrix}$

b) They create the diagonals of parallelograms

c) They form all of the plane $\mathbb{R}^2$

d) both a) and b)

e) both a) and c)

f) both b) and c)

g) all of a), b), and c)

Span of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

a) a point

b) a line in $\mathbb{R}^2$

c) a line in $\mathbb{R}^3$

d) a plane in $\mathbb{R}^2$

e) a plane in $\mathbb{R}^3$

f) a hyperplane in higher dimensions

g) non-linear

Earliest Known Uses of some of the Words of Mathematics: *Linear combination in G. W. Hill, "On the Extension of Delaunay's Method in the Lunar Theory to the General Problem of Planetary Motion," Transactions of the American Mathematical Society, 1(2), Apr. 1900.*

Collections of Vectors in $\mathbb{R}^3$

Notice that $-1\begin{bmatrix}1\\4\\7\end{bmatrix}+2\begin{bmatrix}2\\5\\8\end{bmatrix}=\begin{bmatrix}3\\6\\9\end{bmatrix}$. More generally, what do the collection of column vectors $c_1\begin{bmatrix}1\\4\\7\end{bmatrix}+c_2\begin{bmatrix}2\\5\\8\end{bmatrix}$, for c_1 and c_2 real, have in common geometrically?

a) the line in $\mathbb{R}^3$ connecting the tips of $\begin{bmatrix}1\\4\\7\end{bmatrix}$ and $\begin{bmatrix}2\\5\\8\end{bmatrix}$

b) the plane in $\mathbb{R}^3$ formed by $\begin{bmatrix}1\\4\\7\end{bmatrix}$ and $\begin{bmatrix}2\\5\\8\end{bmatrix}$

c) a non-linear curve or surface

d) none of the above

Extension of 1.3 Question

```
s13extension:=Matrix([[1,-5,b1],[3,-8,b2],[-1,2,b3]]);
ReducedRowEchelonForm(s13extension);
```

outputs the 3x3 identity $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Which are true?

a) $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is never in the span of $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is never a linear combination of $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is never in the plane formed by $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}$

d) all of the above

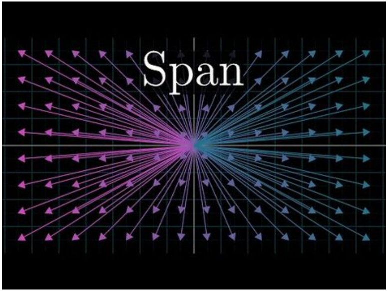
e) none of the above

Span Comparison

Compare the span of the 3 vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, to the span of the 2 vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

a) The spans are the same

b) The spans are different



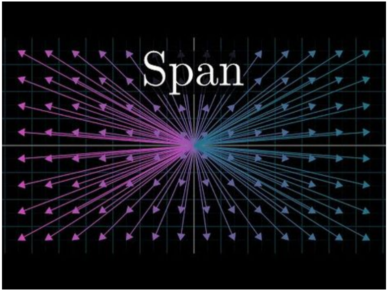
https://i.ytimg.com/vi/k7RM-ot2NWY/0.jpg

Span Comparison

Compare the span of the 3 vectors $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}$, and $\begin{bmatrix}0\\1\end{bmatrix}$, to the span of the 2 vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

a) The spans are the same

b) The spans are different



https://i.ytimg.com/vi/k7RM-ot2NWY/0.jpg

$$\begin{bmatrix}1 & 1 & 0 & b_1\\0 & 1 & 1 & b_2\end{bmatrix} \quad \begin{bmatrix}1 & 0 & b_1\\0 & 1 & b_2\end{bmatrix}$$