1.4 Matrix Equation $A\vec{x} = \vec{b}$

View 1: system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 + 6x_2 - x_3 &= 4 \\ 5x_1 + 10x_2 + x_3 &= 10 \end{aligned}$$

Goal: find all simultaneous solutions

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View 2: vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}$$

Goal: find all $x_1, x_2,$ and x_3 that satisfy this equation.

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Goal: find all $x_1, x_2,$ and x_3 that satisfy this equation.

View 3: matrix-vector equation

$$A\vec{x} = \vec{b}$$

Goal: Find all vectors $\vec{x}$ that satisfy this equation.

Matrix-Vector Products

If A is an $m \times n$ matrix, with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ and if $\vec{x}$ is in $\mathbb{R}^n$, then the *product of A and $\vec{x}$*, denoted by $A\vec{x}$, is the linear combination of the columns of A using the corresponding entries in x as weights. That is,

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

$$\begin{bmatrix} 7 & -3 \\ 2 & -3 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5 \begin{bmatrix} 7 \\ 2 \\ 9 \\ -3 \end{bmatrix} + 4 \begin{bmatrix} -3 \\ -3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 7 + 4(-3) \\ 5 \cdot 2 + 4(-3) \\ 5 \cdot 9 + 4(-6) \\ 5(-3) + 4 \cdot 2 \end{bmatrix}$$

Multiple Representations

$$\begin{aligned} x_1 + 5x_2 - 3x_3 - 4x_4 &= 0 \\ -x_1 - 4x_2 + x_3 + 3x_4 &= 1 \\ -2x_1 - 7x_2 + x_4 &= 0 \end{aligned}$$

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$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & = \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 5 & -3 & -4 & 0 \\ -1 & -4 & 1 & 3 & 1 \\ -2 & -7 & 0 & 1 & 0 \end{array}\right]$$

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$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

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$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

Which Matrix-Vector Product is Impossible?

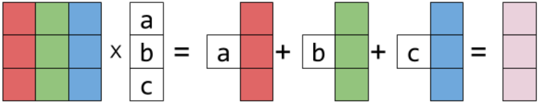
a) $\begin{bmatrix} -4 & 2 & 3 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

b) $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 5 & 3 \\ -1 & 6 & 10 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}$

d) none as all are possible

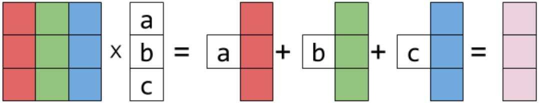
Identity Matrix



https://eli.thegreenplace.net/2015/visualizing-matrix-multiplication-as-a-linear-combination/

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Identity Matrix



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$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\2\\4\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\2\\4\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\2\\4\end{bmatrix}$$

I_n is the $n \times n$ identity matrix—multiplicative identity
1's on the diagonal and 0's everywhere else

Coffee Mixing Representations

	House	Deluxe
Brazil	30%	40%
Columbia	20%	30%
Kenya	20%	20%
Sumatra	30%	10%

The shop has 36 lbs of Brazil roast, 26 lbs of Columbia roast, 20 lbs of Kenya roast, and 18 lbs of Sumatra roast in stock. Can they use up all their coffee making these two blends?

1. Is $\vec{b} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix}$ in the span of $\vec{h} = \begin{bmatrix} 3/10 \\ 2/10 \\ 2/10 \\ 3/10 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 4/10 \\ 3/10 \\ 2/10 \\ 1/10 \end{bmatrix}$?
2. Is $\vec{b}$ a linear combination of $\vec{h}$ and $\vec{d}$?
3. Can we find scalars c_1 and c_2 such that $c_1\vec{h} + c_2\vec{d} = \vec{b}$?

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4. Is the matrix equation $\underbrace{[\vec{h} \ \ \vec{d}]}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{x}} = \vec{b}$ consistent?

Coffee Mixing Representations 1–4

4. Is the matrix equation $\underbrace{[\vec{h} \;\; \vec{d}]}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{x}} = \vec{b}$ consistent?

$$\left[\begin{array}{cc|c} c_1 & c_2 & = \\ 3/10 & 4/10 & 36 \\ 2/10 & 3/10 & 26 \\ 2/10 & 2/10 & 20 \\ \underbrace{3/10}_{\vec{h}} & \underbrace{1/10}_{\vec{d}} & \underbrace{18}_{\vec{b}} \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 40 \\ 0 & 1 & 60 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 40$ and $c_2 = 60$

Yes, we can use up all of the coffee by making 40 pounds of house blend and 60 pounds of deluxe blend.

Coffee Mixing Representations 1–4

1. Is $\vec{b} = \begin{bmatrix} 36 \\ 26 \\ 20 \\ 18 \end{bmatrix}$ in the span of $\vec{h} = \begin{bmatrix} 3/10 \\ 2/10 \\ 2/10 \\ 3/10 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 4/10 \\ 3/10 \\ 2/10 \\ 1/10 \end{bmatrix}$? *Yes*
2. Is $\vec{b}$ a linear combination of $\vec{h}$ and $\vec{d}$? *Yes*
3. Can we find scalars c_1 and c_2 such that $c_1\vec{h} + c_2\vec{d} = \vec{b}$? *Yes*
4. Is the matrix equation $[\vec{h}\ \ \vec{d}] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{b}$ consistent? *Yes*

Coffee Mixing Representations for a Generic Vector

Would we be able to use up any distribution of coffee making just these two blends?

1. Is every $\vec{b}$ in $\mathbb{R}^4$ in the span of $\vec{h} = \begin{bmatrix} 3/10 \\ 2/10 \\ 2/10 \\ 3/10 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 4/10 \\ 3/10 \\ 2/10 \\ 1/10 \end{bmatrix}$?
2. Is every $\vec{b}$ in $\mathbb{R}^4$ a linear combination of $\vec{h}$ and $\vec{d}$?
3. Does $c_1\vec{h} + c_2\vec{d} = \vec{b}$ have a solution for all $\vec{b}$ in $\mathbb{R}^4$?
4. Is $[\vec{h}\ \vec{d}]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{b}$ consistent for every $\vec{b}$ in $\mathbb{R}^4$?

No

Coffee Mixing Representations for a Generic Vector

$$\begin{bmatrix} \underbrace{\begin{matrix} 3/10 \\ 2/10 \\ 1/10 \\ 3/10 \end{matrix}}_{\vec{h}} & \underbrace{\begin{matrix} 4/10 \\ 3/10 \\ 2/10 \\ 1/10 \end{matrix}}_{\vec{d}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} r \\ c \\ k \\ s \end{matrix}}_{\vec{b}} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3/10 & 4/10 & r \\ 2/10 & 3/10 & c \\ 1/10 & 2/10 & k \\ 3/10 & 1/10 & s \end{array}\right] \xrightarrow{\text{GaussianElimination}} \left[\begin{array}{cc|r} 3/10 & 2/5 & r \\ 0 & 1/30 & c - 2r/3 \\ 0 & 0 & k - 2r + 2c \\ 0 & 0 & 0 \end{array}\right]$$

We only use up all the stock when $k - 2r + 2c = 0$, i.e. $-2r + 2c + k + 0s = 0$ equation $\vec{b}$ in the span satisfy

Connections Theorem (Part 1)

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true or they are all false.

1. (matrix equation version) For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
2. (linear combination version) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. (span version) The columns of A span $\mathbb{R}^m$.
4. (systems version) A has a pivot position in every row.

Examples for the Connections Theorem

- Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  (a) Does $A\vec{x} = \vec{b}$ have a solution for each $\vec{b} \in \mathbb{R}^2$?
  (b) Is every vector $\vec{b}$ in $\mathbb{R}^2$ a linear combination of the columns of A?
  (c) Do the columns of A span $\mathbb{R}^2$?

Examples for the Connections Theorem

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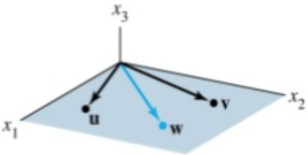
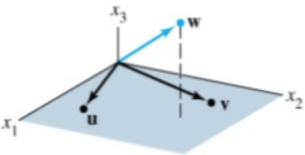
  (b) Is every vector $\vec{b}$ in $\mathbb{R}^2$ a linear combination of the columns of A?
  (c) Do the columns of A span $\mathbb{R}^2$?
- Let $D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
  (a) Does $D\vec{x} = \vec{b}$ have a solution for each $\vec{b}$ in $\mathbb{R}^2$?
  (b) Do the columns of D span $\mathbb{R}^2$?

In or Out of the Plane?

Is $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in the span of $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$?



or



Linear Algebra and Its Applications by David Lay, Steven Lay, and Judi J. McDonald

Multiple Representations

Is $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in the span of $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$?

1. linear combinations

2. a vector equation

3. a matrix-vector product

4. an augmented matrix for a system of equations

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   Does $c_1\vec{u} + c_2\vec{v} = \vec{w}$ have a solution for c_1, c_2?

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   Is $[\vec{u}\ \vec{v}] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{w}$ consistent?

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   Is $[\vec{u}\ \vec{v}] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{w}$ consistent?

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   Is $[\vec{u}\ \vec{v} | \vec{w}]$ consistent?

Back to: In or Out of the Plane?

Is $\vec{w} = \begin{bmatrix}1\\3\\2\end{bmatrix}$ in the span of $\vec{u} = \begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\0\\-5\end{bmatrix}$?

Is the system with augmented matrix $[\vec{u}\ \vec{v}|\vec{w}]$ consistent?

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & -5 & 2 \end{array}\right] \xrightarrow{\text{GaussianElimination}}$$

Back to: In or Out of the Plane?

Is $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in the span of $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$?

Is the system with augmented matrix $[\vec{u}\ \vec{v}|\vec{w}]$ consistent?

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & -5 & 2 \end{array}\right] \xrightarrow{\text{GaussianElimination}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$0 = non - zero\ problem$ $\leftrightarrow$ No solution/inconsistent
$\leftrightarrow$ w is not in the span of $\vec{u}$ and $\vec{v}$

Dot Product View of Matrix-Vector Products

The dot product of two vectors in $\mathbb{R}^n$ is defined as

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = (1)(4) + (2)(5) + (3)(6) = 32$$

The row-vector definition for $A\vec{x}$ has the i^{th} entry in the product as the dot product of the i^{th} row of A with $\vec{x}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} =$$

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$$=$$

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$$= \begin{bmatrix} (1)(1) + (2)(7) + (3)(11) \\ (4)(1) + (5)(7) + (6)(11) \\ (7)(1) + (8)(7) + (9)(11) \end{bmatrix} = \begin{bmatrix} 48 \\ 105 \\ 162 \end{bmatrix}$$

The row-vector rule for computing $A\vec{x}$ says that to get the i^{th} entry in the product $A\vec{x}$ you take the dot product of the i^{th} row of A with $\vec{x}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} \\ \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1) + (2)(7) + (3)(11) \\ (4)(1) + (5)(7) + (6)(11) \\ \underbrace{(7)}_{col.\ 1}(1) + \underbrace{(8)}_{col.\ 2}(7) + \underbrace{(9)}_{col.\ 3}(11) \end{bmatrix}$$

Matrix-Vector Product in 2 ways

Find $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ in two different ways.

Matrix-Vector Product in 2 ways

Find $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ in two different ways.

linear combinations

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

dot products

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} (1)(4) + (0)(-1) + (3)(2) \\ (0)(4) + (5)(-1) + (7)(2) \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

Linear Combinations vs Dot Product

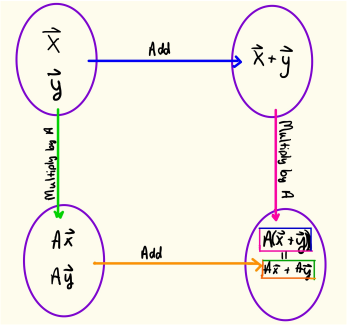
Popular professional algorithms for matrix computations use the programing language Fortran. Fortran stores matrices as a set of columns. The programing language C stores matrices as a collection of rows. Which would be most efficient?

a) Fortran would use the linear combinations while C would use the dot-products.
b) Fortran would use the dot-products while C would use the linear combinations.
c) both languages would use linear combinations.
d) both languages would use the dot products.

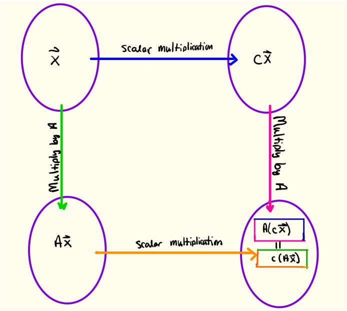
Addition and Scalar Multiplication

If $A_{m \times n}$, $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$, and c is real, then:

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
2. $A(c\vec{x}) = c(A\vec{x})$



Preserves addition



Preserves scalar multiplication

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Linear algebra is often one of the first classes students see that is not procedural. It is full of new mathematical language, abstract and spatial thinking, algebraic arguments, visual arguments, real-life numerical applications, and other analyses that students must internalize in order to succeed. It is also a class where a computer algebra software system is required. Students who did well in earlier classes through short term memorization often struggle in linear algebra.

Christine Andrews-Larson, Jason Siefken, and Rahul Simha (2022). "Report on a US-Canadian Faculty Survey on Undergraduate Linear Algebra: Could Linear Algebra Be an Alternate First Collegiate Math Course?" *Notices of the American Mathematical Society* 69(5) p. 809.

Make it Stick: The Science of Successful Learning by Peter C Brown, Henry L. Roediger III, and Mark A. McDaniel

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left[\begin{array}{cccc} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow[r_3'=-7r_1+r_3]{r_2'=-4r_1+r_2}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix} \xrightarrow[r_3'=-7r_1+r_3]{r_2'=-4r_1+r_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -4\cdot 1+4 & -4\cdot 2+5 & -4\cdot 3+6 & -4\cdot 0+0 \\ -7\cdot 1+7 & -7\cdot 2+8 & -7\cdot 3+9 & -7\cdot 0+0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{bmatrix} \xrightarrow[r_3'=-7r_1+r_3]{r_2'=-4r_1+r_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -4\cdot 1+4 & -4\cdot 2+5 & -4\cdot 3+6 & -4\cdot 0+0 \\ -7\cdot 1+7 & -7\cdot 2+8 & -7\cdot 3+9 & -7\cdot 0+0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\qquad\begin{bmatrix}1&2&3&0\\4&5&6&0\\7&8&9&0\end{bmatrix}\xrightarrow[r_3'=-7r_1+r_3]{r_2'=-4r_1+r_2}$$

$$\begin{bmatrix}1&2&3&0\\-4\cdot1+4&-4\cdot2+5&-4\cdot3+6&-4\cdot0+0\\-7\cdot1+7&-7\cdot2+8&-7\cdot3+9&-7\cdot0+0\end{bmatrix}=$$

$$\begin{bmatrix}1&2&3&0\\0&-3&-6&0\\0&-6&-12&0\end{bmatrix}\xrightarrow{r_3'=-2r_2+r_3}\begin{bmatrix}1&2&3&0\\0&-3&-6&-0\\0&0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \begin{bmatrix}1&2&3&0\\4&5&6&0\\7&8&9&0\end{bmatrix}\xrightarrow[r_3'=-7r_1+r_3]{r_2'=-4r_1+r_2}$$

$$\begin{bmatrix}1&2&3&0\\-4\cdot1+4&-4\cdot2+5&-4\cdot3+6&-4\cdot0+0\\-7\cdot1+7&-7\cdot2+8&-7\cdot3+9&-7\cdot0+0\end{bmatrix}=$$

$$\begin{bmatrix}1&2&3&0\\0&-3&-6&0\\0&-6&-12&0\end{bmatrix}\xrightarrow{r_3'=-2r_2+r_3}\begin{bmatrix}1&2&3&0\\0&-3&-6&-0\\0&0&0&0\end{bmatrix}$$

consistent—yes! x_3 has no pivot so parameter t. Backsubtitute variables attached to pivots in terms of parameters:

row 2: $-3x_2-6x_3=0$ so $-3x_2-6\,t=0$, i.e. $x_2=-2t$

row 1: $x_1+2x_2+3x_3=0$ so $x_1+2(-2t)+3(t)=0$, i.e.

$x_1=-2(-2t)-3t=t$

$$c_1\begin{bmatrix}1\\4\\7\end{bmatrix}+c_2\begin{bmatrix}2\\5\\8\end{bmatrix}+c_3\begin{bmatrix}3\\6\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$ has infinite solutions

Image Credit: Phil B `https://educationcogitation.com/tag/science/`
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