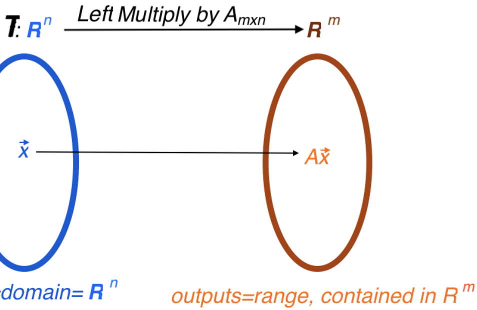
1.8 and 1.9 Linear Transformations

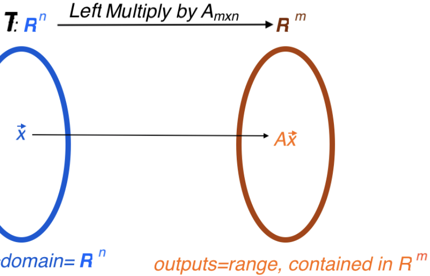


$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n$$

range $T(\vec{x}) = A\vec{x}$ is the span of the columns of A

Function/Linear Transformation Viewpoints

left multiply by A to get $A\vec{x}$ — *$\vec{b}$ is the image* of $\vec{x}$ under the transformation $T(\vec{x}) = A\vec{x}$



solve $A\vec{x} = \vec{b}$ for $\vec{b}$ — find the output $\vec{b}$ in the span of columns of A

solve $A\vec{x} = \vec{b}$ for $\vec{x}$ — find the inputs that mapped to the output $\vec{b}$

Review of Dot Products for Matrix Multiplication

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

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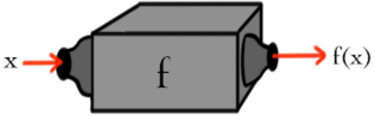
$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \text{row } 1A \end{bmatrix} \cdot \begin{bmatrix} \\ \text{col } 1B \\ \\ \end{bmatrix} & \begin{bmatrix} \text{row } 1A \end{bmatrix} \cdot \begin{bmatrix} \\ \text{col } 2B \\ \\ \end{bmatrix} \\ \begin{bmatrix} \text{row } 2A \end{bmatrix} \cdot \begin{bmatrix} \\ \text{col } 1B \\ \\ \end{bmatrix} & \begin{bmatrix} \text{row } 2A \end{bmatrix} \cdot \begin{bmatrix} \\ \text{col } 2B \\ \\ \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \\ \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix} & \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix}$$

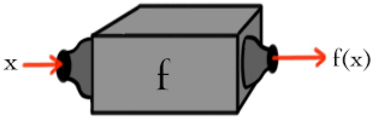
Invertibility or Not in Linear Transformations



https://www.math.ucdavis.edu/~linear/linear-guest.pdf

- A invertible transforms in ways that can be undone, without a loss of information
- A not invertible like $T : \mathbb{R}^3 \to \mathbb{R}^3$ via $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ acts on $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. What is $T(\vec{x})$?

Invertibility or Not in Linear Transformations



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Nonsquare Linear Transformations

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix}$ and the linear transformation $T(\vec{x}) = A\vec{x}$.

Fill in the blanks $T : \underbrace{\mathbb{R}^{\_\_\_}}_{\text{inputs}} \to \underbrace{\mathbb{R}^{\_\_\_}}_{\text{contains outputs}}$

Nonsquare Linear Transformations

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix}$ and the linear transformation $T(\vec{x}) = A\vec{x}$.

Fill in the blanks T : $\underbrace{\mathbb{R}^{\_\_}}_{\text{inputs}} \to \underbrace{\mathbb{R}^{\_\_}}_{\text{contains outputs}}$

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} * \\ * \\ * \end{bmatrix}}_{\textit{input}} = \underbrace{\begin{bmatrix} * \\ * \end{bmatrix}}_{\textit{output or image}}$$

Nonsquare Linear Transformations

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix}$ and the linear transformation $T(\vec{x}) = A\vec{x}$.

Fill in the blanks T : $\underbrace{\mathbb{R}^{\underline{3}}}_{\text{inputs}} \to \underbrace{\mathbb{R}^{\underline{2}}}_{\text{contains outputs}}$

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} * \\ * \\ * \end{bmatrix}}_{\textit{input}} = \underbrace{\begin{bmatrix} * \\ * \end{bmatrix}}_{\textit{output or image}}$$

Vertical Shear

Apply the transformation $T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \vec{x}$ to the vectors

$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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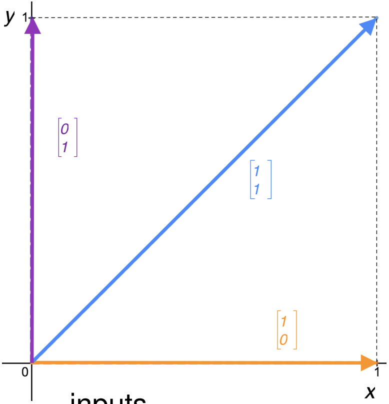
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 \\ 2 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

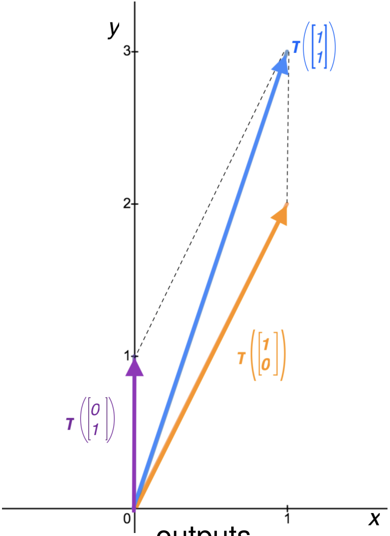
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Plot the vectors in 1 graph and their images in another.
How could we describe T?

Vertical Shear $T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \vec{x}$ *on Unit Square*



inputs



outputs

Vertical Shear

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

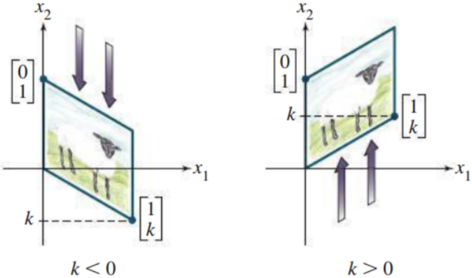
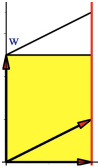


Image from *Linear Algebra and Its Applications* by David Lay, Steven Lay, and Judi McDonald



Images created using VLA Package from *Visual Linear Algebra* by Gene Herman and Michael Pepe

input-output diagram

Projection onto $y = x$

Apply the transformation $T(\vec{x}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x}$ to $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and plot the outputs on 1 graph

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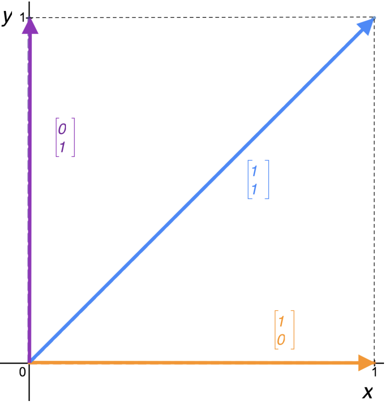
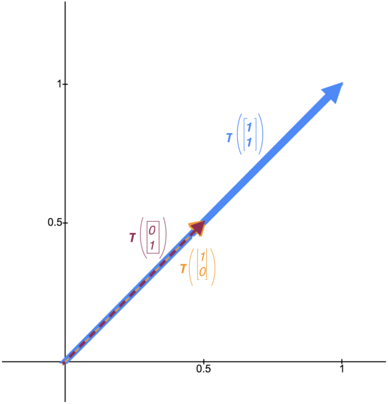
$$
\begin{aligned}
T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}
\end{aligned}
$$

Projection $T(\vec{x}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x}$ *onto* $y = x$



inputs



outputs

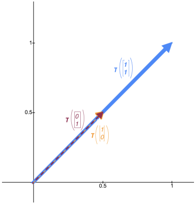
Projection $T(\vec{x}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x}$ onto $y = x$

Is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ in the range of this transformation?

Projection $T(\vec{x}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x}$ *onto* $y = x$

Is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ in the range of this transformation?

geometry:



no: T projects onto $y = x$

but $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is not on the line

algebra: Can we find a solution to $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$?

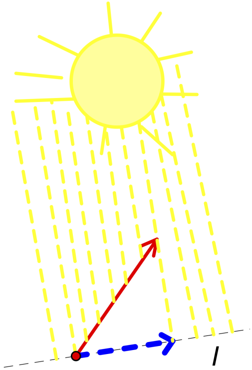
$$\left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -2 \end{array}\right] \xrightarrow{r_2' = -r_1 + r_2} \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -3 \end{array}\right]$$

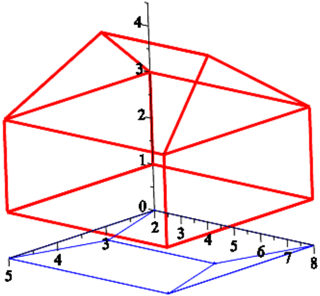
no solution $\rightarrow$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is not in the range of T

Projection

$$\text{Hotel3d}=\begin{bmatrix} 5 & 2 & 2 & 5 & 5 & 5 & 2 & 2 & 3.5 & 3.5 & 2 & 2 & 5 & 3.5 & 3.5 & 5 & 5 & 5 & 2 & 2 & 5 \\ 2 & 2 & 2 & 2 & 2 & 8 & 8 & 8 & 7 & 3 & 2 & 8 & 8 & 7 & 3 & 2 & 8 & 8 & 8 & 2 & 2 \\ 1 & 1 & 3 & 3 & 1 & 1 & 1 & 3 & 4.5 & 4.5 & 3 & 3 & 3 & 4.5 & 4.5 & 3 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Shadow light rays perpendicular to projection make



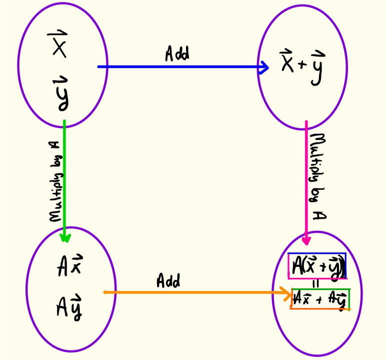


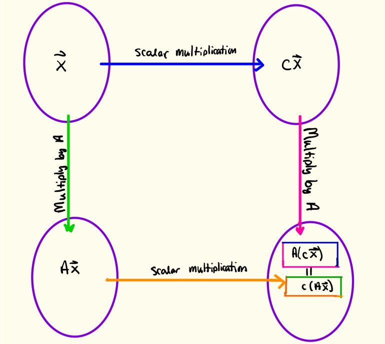
input-output diagram in $\mathbb{R}^2$ input-output diagram in $\mathbb{R}^3$

What Makes these Transformations Linear?

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if

1. $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$ for all $\vec{x}_1, \vec{x}_2$ in $\mathbb{R}^n$
2. $T(c\vec{x}) = cT(\vec{x})$ for all scalars c and all $\vec{x}$ in $\mathbb{R}^n$





Consequences of Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

a) There exists a unique matrix representation $A_{m \times n}$

b) $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n)$

c) $T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}$

d) If we know where the unit x-axis and unit y-axis map to, we know the matrix of the linear transformation.
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$

Consequences of Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

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c) $T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \text{1st column of matrix representation}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \text{2nd column of matrix representation}$$

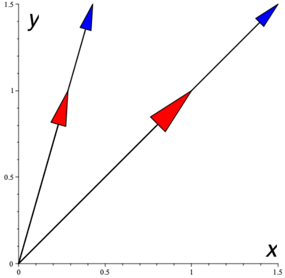
Applying Linearity to Dilation $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

the output of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the 1st column $\begin{bmatrix} a \\ 0 \end{bmatrix}$

the output of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the 2nd column $\begin{bmatrix} 0 \\ a \end{bmatrix}$

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = T(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = xT(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + yT(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) =$$

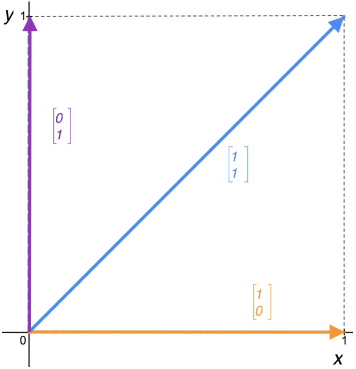
$$x\begin{bmatrix} a \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ a \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}$$

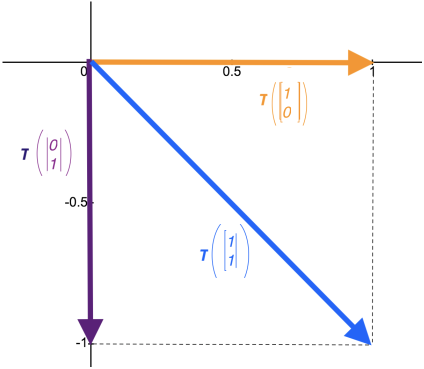


$a > 1$

Reflection across x-axis

What are $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$? Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$.





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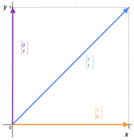
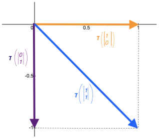




$T(\vec{x}) = A\vec{x}$ for all $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$

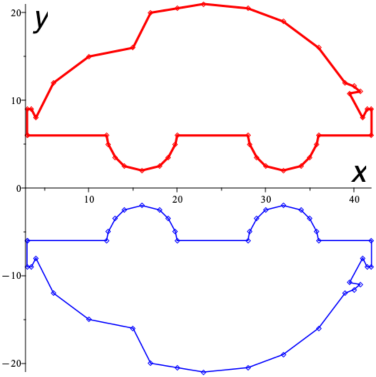
$$= \underbrace{\begin{bmatrix} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\end{bmatrix}}_{\textit{The standard matrix of } T}$$

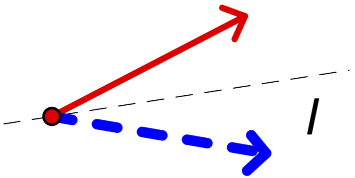
$$= \underbrace{\begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}}_{\textit{The standard matrix of } T}$$

Reflection

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.\text{Car}$$

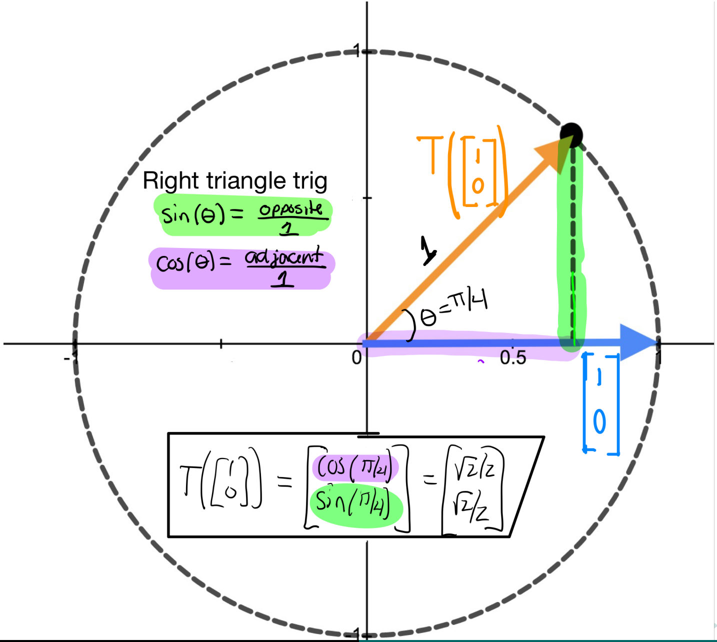
$$\text{Car}=\begin{bmatrix} 5 & 2 & 2 & 5 & 5 & 5 & 2 & 2 & 3.5 & 3.5 & 2 & 2 & 5 & 3.5 & 3.5 & 5 & 5 & 5 & 2 & 2 & 5 \\ 2 & 2 & 2 & 2 & 2 & 8 & 8 & 8 & 7 & 3 & 2 & 8 & 8 & 7 & 3 & 2 & 8 & 8 & 8 & 2 & 2 \end{bmatrix}$$





input-output diagrams

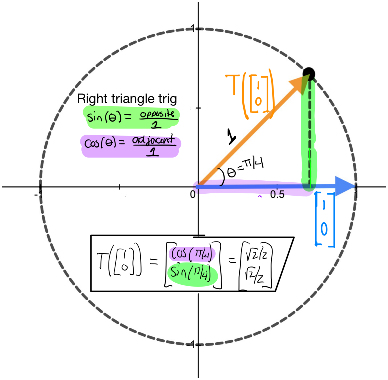
Rotation by $\frac{\pi}{4}$ about $\vec{0}$

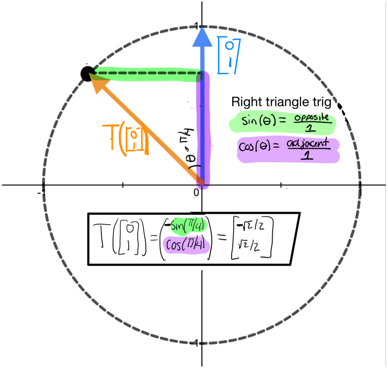


Rotation by $\frac{\pi}{4}$ about $\vec{0}$

If $T(\vec{x})$ rotates $\vec{x}$ in $\mathbb{R}^2$ counter-clockwise by $\frac{\pi}{4}$

$$\left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right] = \begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$$

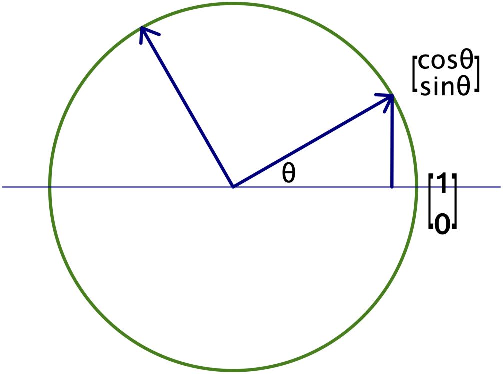




Rotation counterclockwise by θ about $\vec{0}$

The standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector through an angle of θ in the counter-clockwise direction is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



Some Transformations of $\mathbb{R}^2$

Dilation: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ Mixed Dilation: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Horizontal Shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ Vertical Shear: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Projection Matrix: $\begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix}$

Reflection Matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

Rotation Matrix: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Visualization in GeoGebra

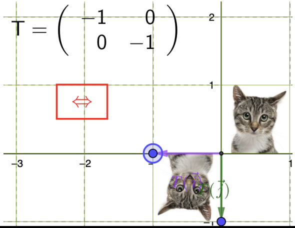
dilation $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

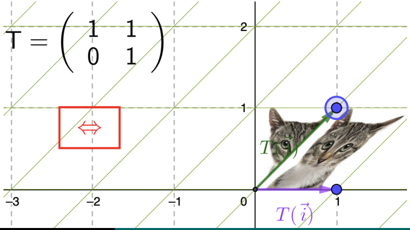
horizontal shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

projection onto the x-axis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

reflection across the y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

rotation by π $\begin{bmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

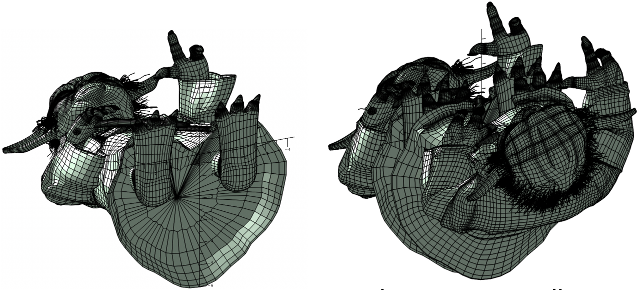




Transformations in $\mathbb{R}^2$ versus $\mathbb{R}^3$

	$\mathbb{R}^2$	$\mathbb{R}^3$
Projection	onto a line	onto a plane
Reflection	about a line	about a plane
Rotation	about the origin	about a line

Reflection about the x-0-z plane $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ z \end{bmatrix}$



input-output diagram

Yoda courtesy of Kecskemeti B. Zoltan and Lucasfilm LTD