2.1 Matrix Operations

- addition
- scalar multiplication
- matrix multiplication
- transpose

What size is this matrix? Matrix([[6,11,-2],[23,31,5]])

a) 2×3

b) 3×2

2.1 Matrix Operations

- addition
- scalar multiplication
- matrix multiplication
- transpose

What size is this matrix? Matrix([[6,11,-2],[23,31,5]])

a) 2×3

b) 3×2

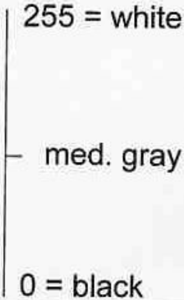
$$A_{2\times 3} = \begin{bmatrix} 6 & 11 & -2 \\ 23 & 31 & 5 \end{bmatrix}$$

Why Consider Linear Algebra of Matrices?

Organization of Data in an Image

value = gray level of pixel at each position

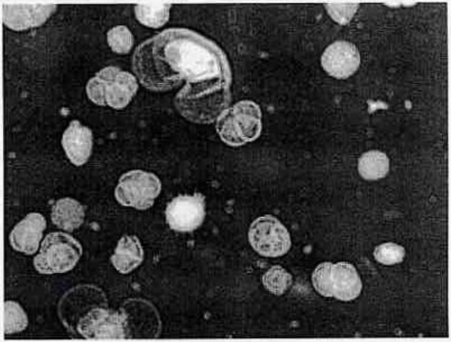
3	3	0	0	72	72	
3	2	0				



Why Consider Linear Algebra of Matrices?

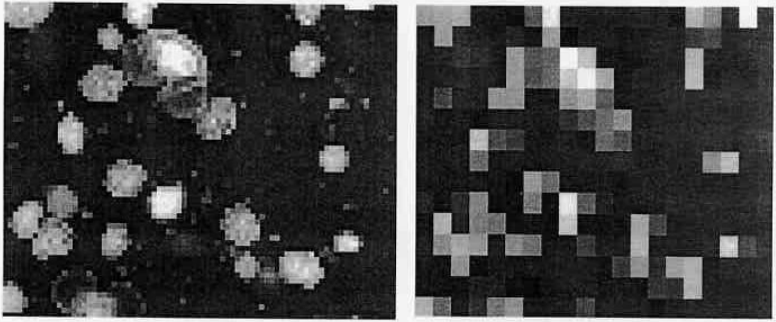
Digital Image of Pollen

- spatial lines = 320 x 240
- brightness levels = 256



Why Consider Linear Algebra of Matrices?

Processed Image of Pollen



spatial lines = 80 x 60

spatial lines = 20x15

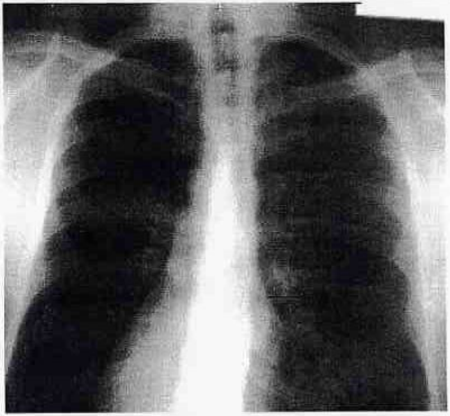
Why Consider Linear Algebra of Matrices?

DIGITAL XRAY IMAGE

Spatial lines:
320 x240

Brightness levels:
256

Lines located at
approximately
x =190 to x= 210
y =150 and y = 170



Why Consider Linear Algebra of Matrices?

Enlarged Section of XRay

Contrast
stretching

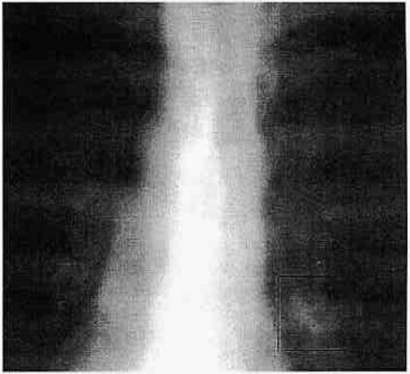
x 1.4

Dark goes darker

Bright goes brighter

Scaling (zoom
in)

X 2



Why Consider Linear Algebra of Matrices?

XRay in Four Level of Brightness

- Identify four shades of gray

3 = white

0 = black

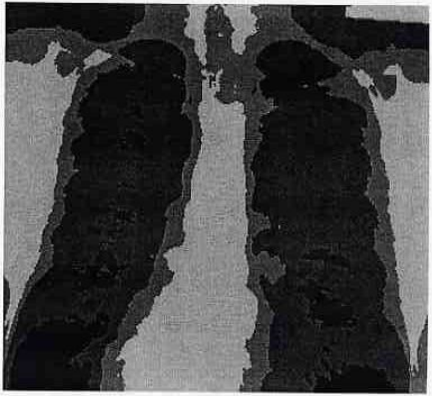


Image Processing Makes Its Mark in Court

Original photograph (*above*) provided to Cognitech for enhancement as part of the prosecution's effort to identify Damian Williams (center of photograph, wearing dark shorts) as the man who attacked Reginald Denny (closest to the open truck door). Cognitech focused on a speck on the arm of the man believed to be Williams, first using an expansion technique to produce an enlargement (above right) of the original photograph, which contains film grain noise but no artifacts. The spot on the subject's arm is still visible but unidentifiable; close inspection also reveals that the subject is wearing two shirts, one over the other (see text). Cognitech's segmentation algorithm was then used to produce the segmentation (*near right*), from which the segmentation map (*far right*) was reconstructed. The image shown is consistent with the rose tattoo on Williams's arm.

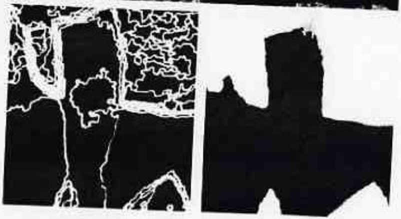


Image Processing Makes Its Mark in Court. December 1993 SIAM News

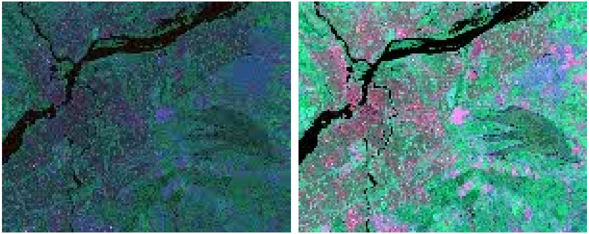
Scalar Multiplication of a Matrix

Let $A = \begin{bmatrix} 4 & 6 \\ 20 & 7 \end{bmatrix}$. What is $5A$?

a) $\begin{bmatrix} 9 & 11 \\ 25 & 12 \end{bmatrix}$

b) $\begin{bmatrix} 20 & 30 \\ 100 & 35 \end{bmatrix}$

c) other



before and after linear stretch

Image credit: https://www.nrcan.gc.ca/earth-sciences/geomatics/satellite-imagery-air-photos/satellite-imagery-products/educational-resources/9389

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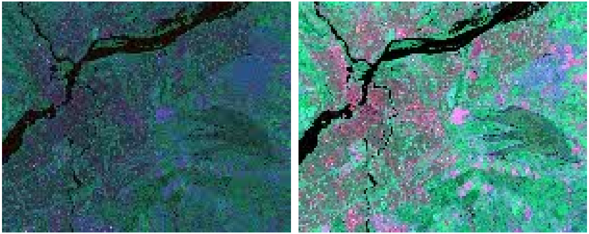
b) $\begin{bmatrix} 20 & 30 \\ 100 & 35 \end{bmatrix}$

c) other



before and after linear stretch

Image credit: `https://www.nrcan.gc.ca/earth-sciences/geomatics/satellite-imagery-air-photos/satellite-imagery-products/educational-resources/9389`

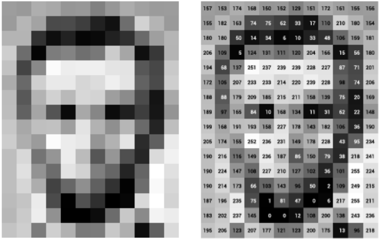
$$cA = c[a_{ij}] = [ca_{ij}]$$

Matrix Addition

Let $A = \begin{bmatrix} 4 & 6 \\ 20 & 24 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ What is $A + B$?

a) $\begin{bmatrix} 6 & 11 \\ 23 & 31 \end{bmatrix}$

b) $\begin{bmatrix} 4 & 6 & 2 & 5 \\ 20 & 24 & 3 & 7 \end{bmatrix}$



157	153	174	168	150	152	129	151	172	161	155	156
155	182	163	74	75	62	33	17	110	210	180	154
180	180	50	14	34	6	10	33	48	106	159	181
206	109	5	124	131	111	120	204	166	15	56	180
194	68	137	251	237	239	239	228	227	87	71	201
172	105	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234
190	216	116	149	236	187	86	150	79	38	218	241
190	224	147	108	227	210	127	102	36	101	255	224
190	214	173	66	103	143	96	50	2	109	249	215
187	196	235	75	1	81	47	0	6	217	255	211
183	202	237	145	0	0	12	108	200	138	243	236
195	206	123	207	177	121	123	200	175	13	96	218

Image credit: Zach Lieberman

https://openframeworks.cc/ofBook/chapters/image_processing_computer_vision.html

Matrix Addition

Let $A = \begin{bmatrix} 4 & 6 \\ 20 & 24 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ What is $A + B$?

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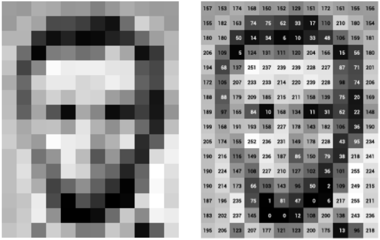

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172	105	207	233	233	214	220	239	228	98	74	206
188	88	179	209	185	215	211	158	139	75	20	169
189	97	165	84	10	168	134	11	31	62	22	148
199	168	191	193	158	227	178	143	182	106	36	190
205	174	155	252	236	231	149	178	228	43	95	234
190	216	116	149	236	187	86	150	79	38	218	241
190	224	147	108	227	210	127	102	36	101	255	224
190	214	173	66	103	143	96	50	2	109	249	215
187	196	235	75	1	81	47	0	6	217	255	211
183	202	237	145	0	0	12	108	200	138	243	236
195	206	123	207	177	121	123	200	175	13	96	218

Image credit: Zach Lieberman

https://openframeworks.cc/ofBook/chapters/image_processing_computer_vision.html

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

Review: Matrix-Vector Multiplication in 2 Ways

linear combinations

$$\begin{bmatrix}1 & 0 & 3\\0 & 5 & 7\end{bmatrix}\begin{bmatrix}4\\-1\\2\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\5\end{bmatrix} + 2\begin{bmatrix}3\\7\end{bmatrix} = \begin{bmatrix}10\\9\end{bmatrix}$$

dot products

$$\begin{bmatrix}1 & 0 & 3\\0 & 5 & 7\end{bmatrix}\begin{bmatrix}4\\-1\\2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & 0 & 3\end{bmatrix}\cdot\begin{bmatrix}4\\-1\\2\end{bmatrix}\\ \begin{bmatrix}0 & 5 & 7\end{bmatrix}\cdot\begin{bmatrix}4\\-1\\2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}(1)(4)+(0)(-1)+(3)(2)\\(0)(4)+(5)(-1)+(7)(2)\end{bmatrix}$$

$$= \begin{bmatrix}10\\9\end{bmatrix}$$

Matrix Multiplication: Match the Units

A fruit grower raises two crops, which are shipped to three outlets.

The number of units of product i that is shipped to outlet j is represented by b_{ij} in the matrix $B = \begin{bmatrix} 100 & 75 & 75 \\ 125 & 150 & 100 \end{bmatrix}$

The profit of one unit of product i is represented by a_{1i} in the matrix $A = \begin{bmatrix} \$3.75 & \$7.00 \end{bmatrix}$

Matrix Multiplication: Match the Units

A fruit grower raises two crops, which are shipped to three outlets.

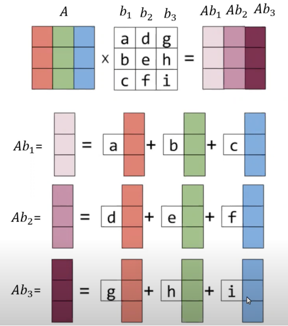
The number of units of product i that is shipped to outlet j is represented by b_{ij} in the matrix $B = \begin{bmatrix} 100 & 75 & 75 \\ 125 & 150 & 100 \end{bmatrix}$

The profit of one unit of product i is represented by a_{1i} in the matrix $A = \begin{bmatrix} \$3.75 & \$7.00 \end{bmatrix}$

$$A\text{col}1B = \begin{bmatrix} \$3.75 & \$7.00 \end{bmatrix} \begin{bmatrix} 100 \\ 125 \end{bmatrix} = 3.75 \cdot 100 + 7 \cdot 125$$

Multiplication: A Times Each Column of B method:

$A_{1\times 2}B_{2\times 3}$ is a 1×3 matrix



$$A \qquad\qquad B$$

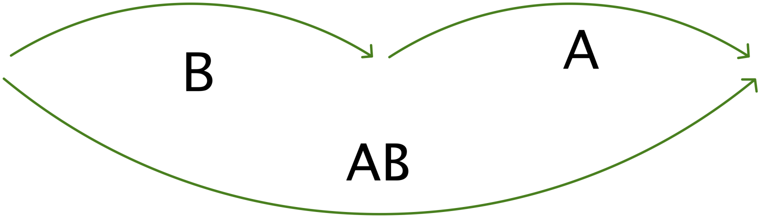
$$= \begin{bmatrix} \$3.75 & \$7.00 \end{bmatrix} \begin{bmatrix} 100 & 75 & 75 \\ 125 & 150 & 100 \end{bmatrix}$$

$$= \left[A \begin{bmatrix} 100 \\ 125 \end{bmatrix} \; A \begin{bmatrix} 75 \\ 150 \end{bmatrix} \; A \begin{bmatrix} 75 \\ 100 \end{bmatrix} \right]$$

`https://www.youtube.com/watch?v=MeiOnxLToRY` adapted from `https://eli.thegreenplace.net/2015/visualizing-matrix-multiplication-as-a-linear-combination/`

$$[3.75 \cdot 100 + 7 \cdot 125 \qquad 3.75 \cdot 75 + 7 \cdot 150 \qquad 3.75 \cdot 75 + 7 \cdot 100]$$

Multiplication of Matrices: Composition



Matrix Multiplication: 2 Methods

Columns of B method: $AB = \begin{bmatrix} A.\text{col}1B & \dots & A.\text{col}nB \end{bmatrix}$

When is multiplication defined? $A_{m\times n}B_{n\times o} = AB_{m\times o}$

Row-column dot product method:

$$AB_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj} = [\text{row}iA] \cdot [\text{column}jB]$$

To obtain the ijth entry of AB we take the ith row of A and the jth column of B, and perform the dot product (line them up, multiply corresponding entries, and add)

$$\begin{bmatrix} \text{row}1A \cdot \text{col}1B & \text{row}1A \cdot \text{col}2B & \dots \\ \text{row}2A \cdot \text{col}1B & \text{row}2A \cdot \text{col}2B & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Matrix Multiplication: 2 Methods

Columns of *B* method: $AB = \begin{bmatrix} A.\text{col}1B & \ldots & A.\text{col}nB \end{bmatrix}$

multiplication of 2 matrices = multiply A by column vectors of B

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6 & 7\\8 & 9 & 10\end{bmatrix} = \left[\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix} \quad \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}6\\9\end{bmatrix} \quad \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}7\\10\end{bmatrix}\right]$$

Do the next step to use the linear combinations of the columns of A using the weights from the cols of B, or the dot products of the rows of A with B (like in section 1.4)

$=$

Matrix Multiplication: 2 Methods

Columns of B method: $AB = \begin{bmatrix} A.\text{col}1B & \dots & A.\text{col}nB \end{bmatrix}$

multiplication of 2 matrices = multiply A by column vectors of B

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix} \end{bmatrix}$$

Do the next step to use the linear combinations of the columns of A using the weights from the cols of B, or the dot products of the rows of A with B (like in section 1.4)

$$= \begin{bmatrix} 1\cdot 5 + 2\cdot 8 & 1\cdot 6 + 2\cdot 9 & 1\cdot 7 + 2\cdot 10 \\ 3\cdot 5 + 4\cdot 8 & 3\cdot 6 + 4\cdot 9 & 3\cdot 7 + 4\cdot 10 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \text{col}1B & \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \text{col}2B & \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \text{col}3B \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \text{col}1B & \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \text{col}2B & \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \text{col}3B \end{bmatrix}$$

$$= \begin{bmatrix} \text{row}1A \cdot \text{col}1B & \text{row}1A \cdot \text{col}2B & \text{row}1A \cdot \text{col}3B \\ \text{row}2A \cdot \text{col}1B & \text{row}2A \cdot \text{col}2B & \text{row}1A \cdot \text{col}3B \end{bmatrix}$$

Matrix Multiplication with Identity and 0 Matrix

A times columns of *B* method for $I_{3\times 3}$ times my favorite matrix

$$\left[\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\4\\7\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2\\5\\8\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3\\6\\9\end{bmatrix}\right]$$

row-column dot product method

$$\begin{bmatrix}\begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}1\\4\\7\end{bmatrix} & \begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}2\\5\\8\end{bmatrix} & \begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix}\\ \begin{bmatrix}0&1&0\end{bmatrix}\cdot\begin{bmatrix}1\\4\\7\end{bmatrix} & \begin{bmatrix}0&1&0\end{bmatrix}\cdot\begin{bmatrix}2\\5\\8\end{bmatrix} & \begin{bmatrix}0&1&0\end{bmatrix}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix}\\ \begin{bmatrix}0&0&1\end{bmatrix}\cdot\begin{bmatrix}1\\4\\7\end{bmatrix} & \begin{bmatrix}0&0&1\end{bmatrix}\cdot\begin{bmatrix}2\\5\\8\end{bmatrix} & \begin{bmatrix}0&0&1\end{bmatrix}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=$$

Matrix Multiplication with Identity and 0 Matrix

A times columns of B method for $I_{3\times 3}$ times my favorite matrix

$$\left[\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\4\\7\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2\\5\\8\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3\\6\\9\end{bmatrix}\right]$$

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$$\begin{bmatrix}\begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}1\\4\\7\end{bmatrix} & \begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}2\\5\\8\end{bmatrix} & \begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix}\\ \begin{bmatrix}0&1&0\end{bmatrix}\cdot\begin{bmatrix}1\\4\\7\end{bmatrix} & \begin{bmatrix}0&1&0\end{bmatrix}\cdot\begin{bmatrix}2\\5\\8\end{bmatrix} & \begin{bmatrix}0&1&0\end{bmatrix}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix}\\ \begin{bmatrix}0&0&1\end{bmatrix}\cdot\begin{bmatrix}1\\4\\7\end{bmatrix} & \begin{bmatrix}0&0&1\end{bmatrix}\cdot\begin{bmatrix}2\\5\\8\end{bmatrix} & \begin{bmatrix}0&0&1\end{bmatrix}\cdot\begin{bmatrix}3\\6\\9\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\quad\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Matrix Multiplication Might Reveal Column Dependency

A times columns of *B* method

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 & 1 \cdot 1 + 2 \cdot -2 + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot 0 + 6 \cdot 0 & 4 \cdot 1 + 5 \cdot -2 + 6 \cdot 1 \\ 7 \cdot 1 + 8 \cdot 0 + 9 \cdot 0 & 7 \cdot 1 + 8 \cdot -2 + 9 \cdot 1 \end{bmatrix}$$

Matrix Multiplication Might Reveal Column Dependency

A times columns of *B* method

$$AB = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -2\\0 & 1\end{bmatrix} = \left[\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}\right]$$

$$= \begin{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}1 & 2 & 3\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}\\ \begin{bmatrix}4 & 5 & 6\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}4 & 5 & 6\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}\\ \begin{bmatrix}7 & 8 & 9\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}7 & 8 & 9\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}1\cdot 1+2\cdot 0+3\cdot 0 & 1\cdot 1+2\cdot -2+3\cdot 1\\4\cdot 1+5\cdot 0+6\cdot 0 & 4\cdot 1+5\cdot -2+6\cdot 1\\7\cdot 1+8\cdot 0+9\cdot 0 & 7\cdot 1+8\cdot -2+9\cdot 1\end{bmatrix}$$

AB has a column of 0s but *B* had no column of 0s, so we have a nontrivial solution to $A\vec{x} = \vec{0}$ via the column that gave us the 0s:

$$AB = \begin{bmatrix}1 & 0\\4 & 0\\7 & 0\end{bmatrix} \text{ and } Ab_2 = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

so the columns of *A* are not linearly independent

Matrix Multiplication with a Generic Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} \text{row1left matrix} \cdot \text{col1right matrix} & \text{row1left matrix} \cdot \text{col2right matrix} \\ \text{row2left matrix} \cdot \text{col1right matrix} & \text{row2left matrix} \cdot \text{col2right matrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ \begin{bmatrix} c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} c & d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a+b & 2a+2b \\ c+d & 2c+2d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

Matrix Multiplication with a Generic Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} \text{row1left matrix} \cdot \text{col1right matrix} & \text{row1left matrix} \cdot \text{col2right matrix} \\ \text{row2left matrix} \cdot \text{col1right matrix} & \text{row2left matrix} \cdot \text{col2right matrix} \end{bmatrix}$$

$$\begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ \begin{bmatrix} c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} c & d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a+b & 2a+2b \\ c+d & 2c+2d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ c \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ c \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ a+2c & b+2d \end{bmatrix}$$

Matrix Multiplication with a Generic Matrix

There exists a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

a) there is exactly 1 matrix A that works
b) there are infinitely many matrices A that work
c) there are no matrices that work
d) none of the above

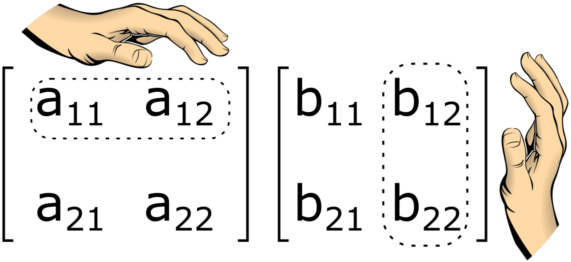


Image credit: Guy vandegrift

https://commons.wikimedia.org/wiki/File:Hands_matrix_multiplication.svg

Sales Algebra

You have a business and May sales are 4 grey tables, 6 white tables, 20 grey chairs, and 24 white chairs, which is represented by the matrix $M = \begin{bmatrix} 4 & 6 \\ 20 & 24 \end{bmatrix}$ where the first row is tables, the second row is chairs, the first column is grey items, and the second column is white items. Your June sales are given by the matrix $J = \begin{bmatrix} 6 & 8 \\ 22 & 32 \end{bmatrix}$. What matrix operations would make sense in real life in this scenario?

a) $M + J$

b) $J - M$

c) $1.2J$

d) MJ

e) all of the above

f) 2 of the above

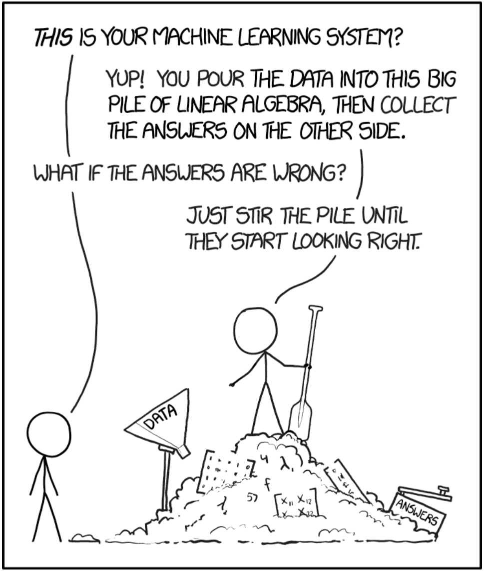
g) 3 of the above

Matrix Multiplication Algebraic Properties

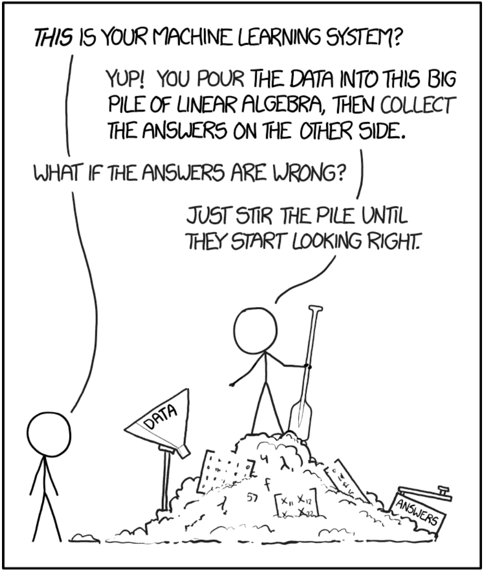
- $A(BC) = (AB)C$
  associative under multiplication
- $A(B + C) = AB + AC$
  left distributive law
- $(B + C)A = BA + CA$
  right distributive law
- $c(AB) = (cA)B = A(cB)$
  scalar multiplication of the product of matrices
- $I_{m\times m}A = A = AI_{n\times n}$
  matrix multiplication identity

Matrix multiplication is not commutative as $AB \neq BA$

Evaluate the statement: If $A\vec{x} = \vec{0}$, then is $C(A\vec{x}) = \vec{0}C$?



Evaluate the statement: If $A\vec{x} = \vec{0}$, then is $C(A\vec{x}) = \vec{0}C$?



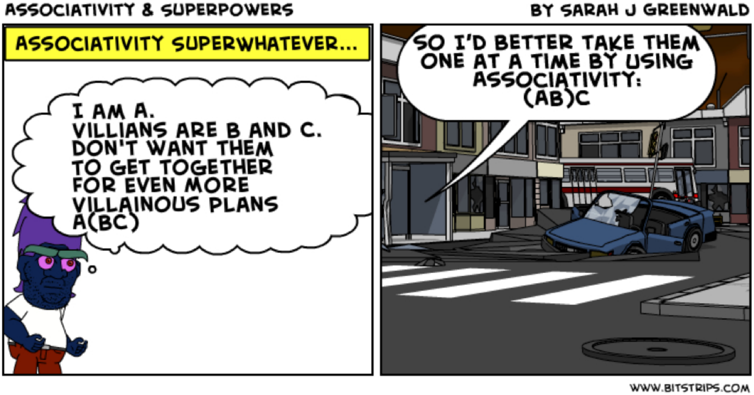
https://imgs.xkcd.com/comics/machine_learning_2x.png

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Applications of Matrix Multiplication

- Hill cipher
  $A_{n\times n}$ [original message]$_{n\times p}$ = [coded message]$_{n\times p}$
  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 21 \\ 19 & 5 \end{bmatrix} = \begin{bmatrix} -19 & 37 \\ -19 & 16 \end{bmatrix}$$
$$\begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 19 \end{bmatrix} & \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 21 \\ 5 \end{bmatrix} \\ \begin{bmatrix} c & d \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 19 \end{bmatrix} & \begin{bmatrix} c & d \end{bmatrix} \cdot \begin{bmatrix} 21 \\ 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0a+19b & 21a+5b \\ 0c+19d & 21c+5d \end{bmatrix}$$
  $$\begin{aligned} 0a+19b &= -19 \\ 21a+5b &= 37 \\ 0c+19d &= -19 \\ 21c+5d &= 16 \end{aligned}$$
  $$\begin{matrix} a & b & c & d & = \end{matrix}$$
  $$\begin{bmatrix} 0 & 19 & 0 & 0 & -19 \\ 21 & 5 & 0 & 0 & 37 \\ 0 & 0 & 0 & 19 & -19 \\ 0 & 0 & 21 & 5 & 16 \end{bmatrix}$$
- Digital images
  *AB*Yoda

The Power of Associativity



digital images, theoretical contexts and spacetime and relativity

If $CA = I_{n\times n}$ and we want to solve $A\vec{x} = \vec{b}$ then
multiply C on the left side: $C(A\vec{x}) = C(\vec{b})$
apply associativity: $(CA)\vec{x} = C\vec{b}$
substitute our given $CA = I_{n\times n}$: $I_{n\times n}\vec{x} = C\vec{b}$
matrix multiplication identity: $\vec{x} = C\vec{b}$

Transpose

$a_{ij} \to a_{ji}$

Transpose

$a_{ij} \to a_{ji}$

- algebraic properties like $(AB)^T = B^T A^T$
- least squares estimates, such as in linear regression
- data given as rows (like Yoda)
- $v^T g_{ij} v$
  Minkowski Spacetime Model

$$\begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} t \\ -x \\ -y \\ -z \end{bmatrix}$$

$$= [t - x^2 - y^2 - z^2]$$

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$