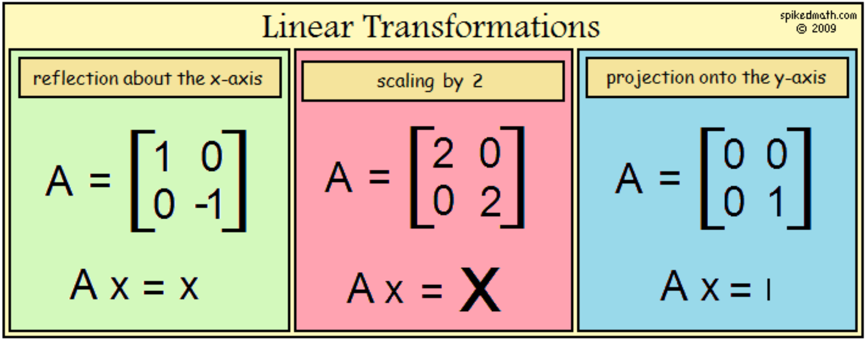
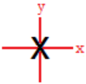
Applications of Linear Transformations

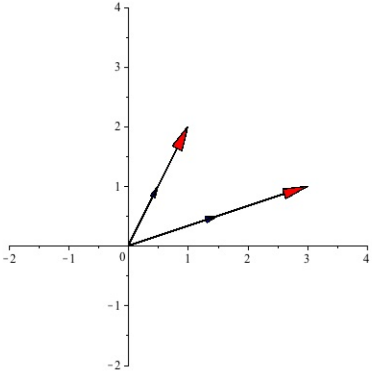
2.7 Applications to Computer Graphics & Hill Cipher





http://spikedmath.com/comics/109-linear-transformations.png

Input red and output blue. What is the name and matrix?



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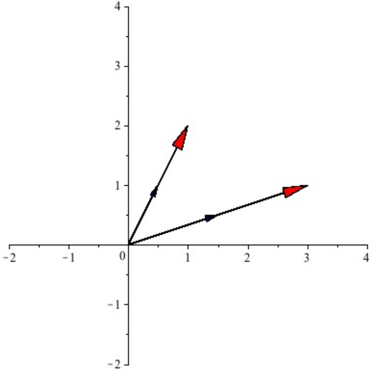
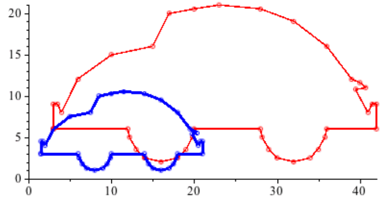
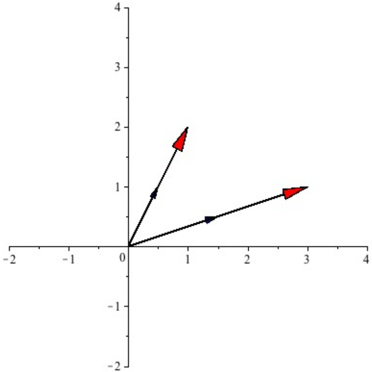


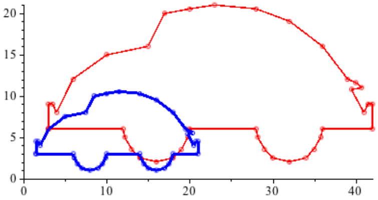


Images created using VLA Package from *Visual Linear Algebra* by Herman and Pepe

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

Input red and output blue. What is the name and matrix?



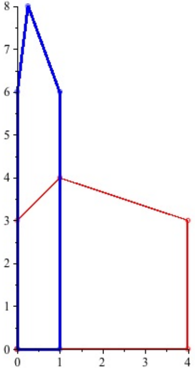


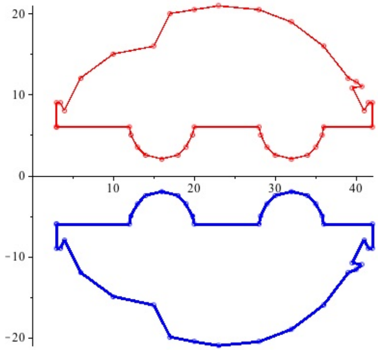
Images created using VLA Package from *Visual Linear Algebra* by Herman and Pepe

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

If we know where the unit x-axis and unit y-axis map to, we know the matrix of the linear transformation.

Input red and output blue. What are the names and matrices?





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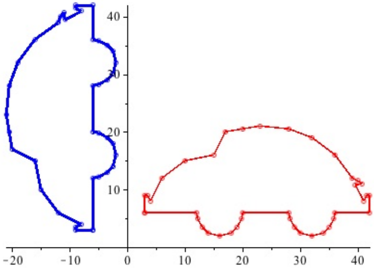
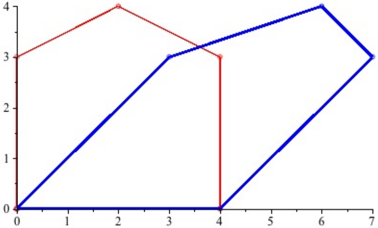
Math isn't important for programming and other hilarious jokes you can tell yourself

Input red and output blue. What are the names and matrices?





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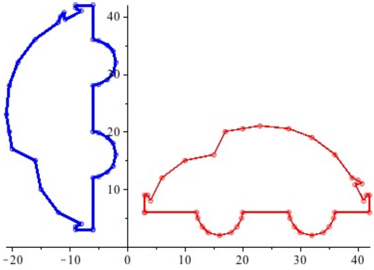
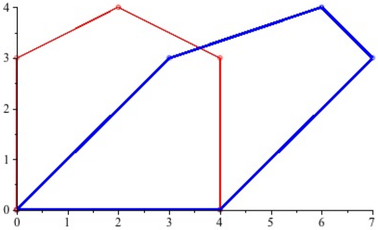


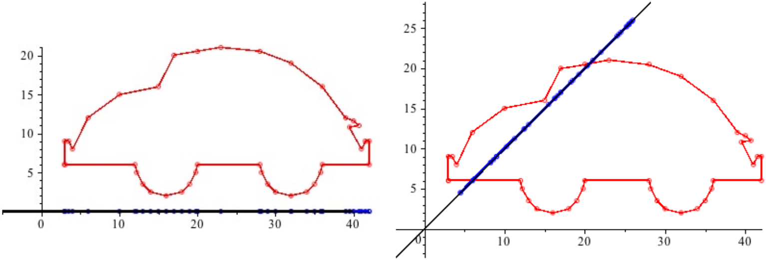


Images created using VLA Package from *Visual Linear Algebra* by Herman and Pepe


sheep

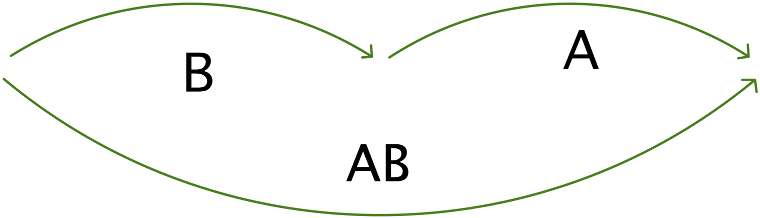

sheared sheep

Input red and output blue. What are the names and matrices?



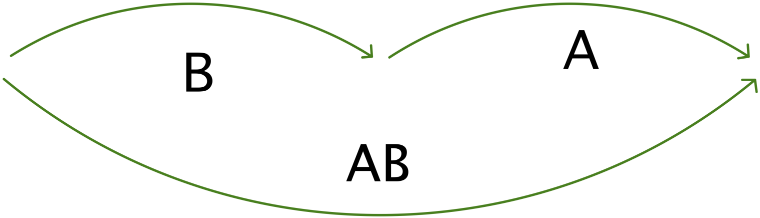
Images created using VLA Package from *Visual Linear Algebra* by Herman and Pepe

Composite Transformations



ABHouse

Composite Transformations



ABHouse

http://cs.appstate.edu/~sjg/class/2240/guess/images/guesstransformation_6.gif

Movie created using VLA Package from *Visual Linear Algebra* by Herman and Pepe

Transformations of $\mathbb{R}^2$

Rotation Matrix: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Dilation Matrix: $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

Horizontal Shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ Vertical Shear: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Projection Matrix: $\begin{bmatrix} \cos(\theta)^2 & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \sin(\theta)^2 \end{bmatrix}$

Reflection Matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

- What important transformation is missing?

Translations and Homogeneous Coordinates

$$\text{translation matrix?} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \end{bmatrix}$$

Translations and Homogeneous Coordinates

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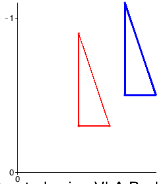
$$\text{translation: } \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ triangle: } \begin{bmatrix} 4 & 4 & 6 & 4 \\ 3 & 9 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



Created using VLA Package from *Visual Linear Algebra* by Herman and Pepe

Rotate a Triangle About its Top Vertex

Step 1. Translate the triangle so the top vertex $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$ is at $\vec{0}$.

Step 2. Rotate the translated figure 45° about $\vec{0}$.

Step 3. Translate back from $\vec{0}$ to where we started at $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$

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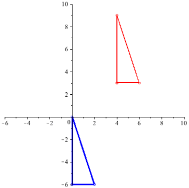
$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) & 0 \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix} \text{Triangle}$$



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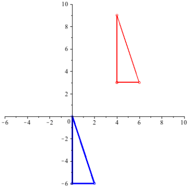

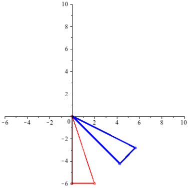

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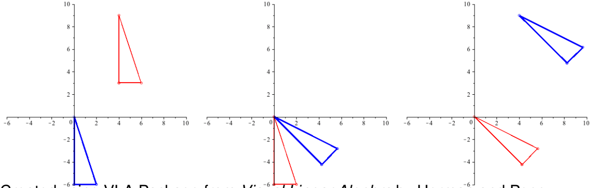

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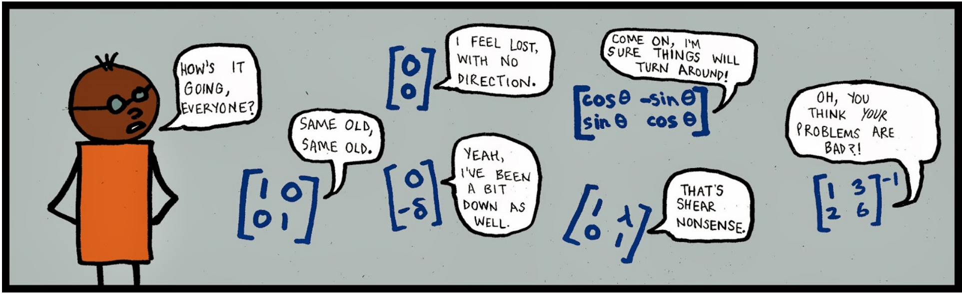
http://cs.appstate.edu/~sjg/class/2240/c4s5short/videos/c4s5shortb_31.gif
http://cs.appstate.edu/~sjg/class/2240/c4s5short/videos/c4s5shortb_32.gif

Which of the following are true about linear transformations?

a) points go in as column vectors, and the first column of the transformation represents the output of the unit x-axis
b) we must use homogeneous coordinates (higher dimensional coordinate=1) if we want to use translations
c) they compose like functions with the first transformation on the right
d) all of the above
e) two of the above



https://pbs.twimg.com/media/BhKmT5RIYAA4iU6.jpg:large

To rotate a figure $\begin{bmatrix} x_1 & \dots & x_p \\ y_1 & \dots & y_p \\ 1 & \dots & 1 \end{bmatrix}$ about a point (-2,3) we can:

a)
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & \dots & x_p \\ y_1 & \dots & y_p \\ 1 & \dots & 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & \dots & x_p \\ y_1 & \dots & y_p \\ 1 & \dots & 1 \end{bmatrix}$$

c) $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & \dots & x_p \\ y_1 & \dots & y_p \\ 1 & \dots & 1 \end{bmatrix}$

d) more than one of the above

e) none of the above

Making a Car Movie

How can we keep a car on the road?

Making a Car Movie

How can we keep a car on the road?

$$\begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix} \text{ car}$$

http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_14.gif

How can we point the car in the direction of motion?

Making a Car Movie

How can we keep a car on the road?

$$\begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix} \text{ car}$$

http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_14.gif

How can we point the car in the direction of motion?

$$\begin{bmatrix} \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ \frac{2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ car}$$

http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_28.gif

Making a Car Movie

How can we keep a car on the road?

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http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_14.gif

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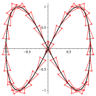
http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_28.gif

$$\begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ \frac{2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ car}$$



VLA Package from *Visual Linear Algebra* by Herman and Pepe

Order and Unit Norm Matters!

$$\begin{bmatrix} \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ \frac{2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix} \text{car}$$

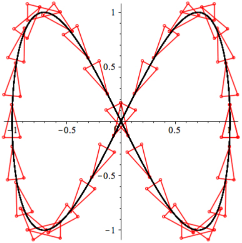
http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_29.gif

$$\begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(t) & -2\cos(2t) & 0 \\ 2\cos(2t) & -\sin(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{car}$$

http://cs.appstate.edu/~sjg/class/2240/c4s5racetrack/videos/c4s5racetrack_31.gif

To turn a car so that it points in the direction of motion, we

a) define a unit vector in the direction of the velocity (tangent) of the curve by dividing by its length/norm so it won't change the size of the car
b) create an orthogonal vector to pair with it in a rotation matrix by creating a vector on a line with negative reciprocal slope (swap *x* and *y* and introduce a negative sign)
c) both of the above



VLA Package from *Visual Linear Algebra* by Herman and Pepe

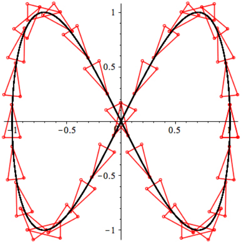
To keep a car on a curved race track, we can perform the appropriate matrix operations in the following order

a) $\begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ \frac{2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ car

b) $\begin{bmatrix} \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ \frac{2\cos(2t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & \frac{-\sin(t)}{\sqrt{\sin^2(t)+4cos^2(2t)}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cos(t) \\ 0 & 1 & \sin(2t) \\ 0 & 0 & 1 \end{bmatrix}$ car

c) both of the above

d) none of the above



VLA Package from *Visual Linear Algebra* by Herman and Pepe

Linear Transformations of $\mathbb{R}^3$ in Homogeneous Coords

$$\begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) & 0 & 0 \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{spaceship}$$

`http://cs.appstate.edu/~sjg/class/2240/c4s5short/videos/c4s5shortb_51.gif`

Created using VLA Package from *Visual Linear Algebra* by Herman and Pepe

Yoda from Star Wars

data from Kecskemeti B. Zoltan (Lucasfilm LTD)

1 of 4667 in file 1: $\begin{bmatrix} -1.355884 & 12.452657 & 0.927687 \\ -1.355386 & 12.449140 & 0.928006 \\ -1.356376 & 12.450524 & 0.924873 \end{bmatrix}$

1 of 29460 in file 2: $\begin{bmatrix} -1.355386 & 12.449140 & 0.928006 \\ -1.355884 & 12.452657 & 0.927687 \\ -1.530717 & 12.421906 & 0.980687 \\ -1.529826 & 12.418444 & 0.980901 \end{bmatrix}$

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$(Rot.Yoda^T)^T = Yoda.Rot^T$ via $(AB)^T = B^T A^T$

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$(Rot.Yoda^T)^T = Yoda.Rot^T$ via $(AB)^T = B^T A^T$

$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}^T =$$

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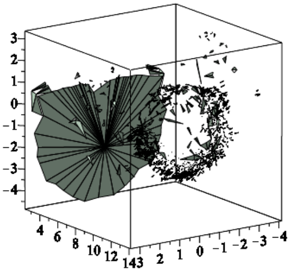
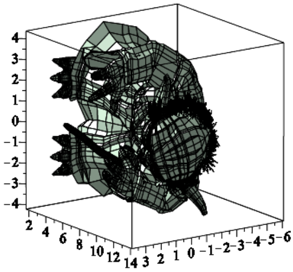
$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}^T = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

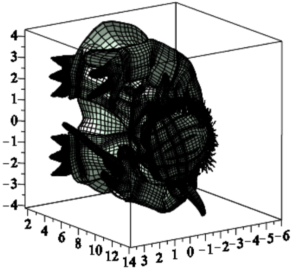
Yoda

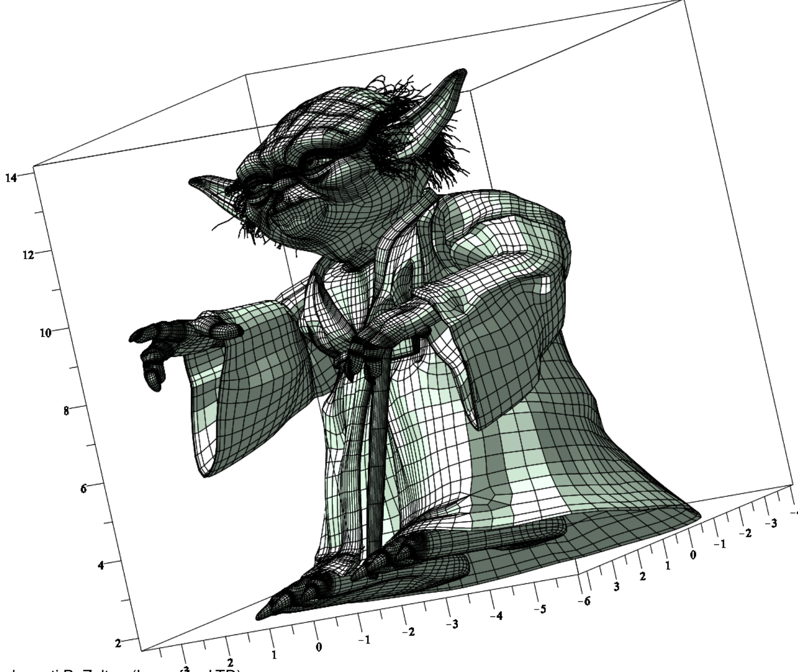


YodaRotated



YodaRotated





data from Kecskemeti B. Zoltan (Lucasfilm LTD)

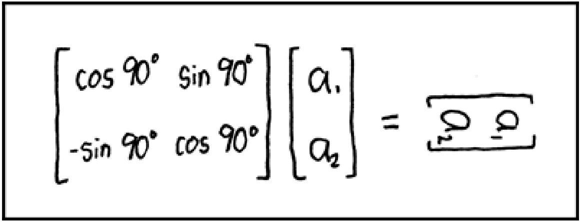
To rotate Yoda, who was given in row vectors as opposed to column vectors, we made use of

a) a matrix that rotates about a line in 3-space

b) $(AB)^T = B^T A^T$

c) both of the above

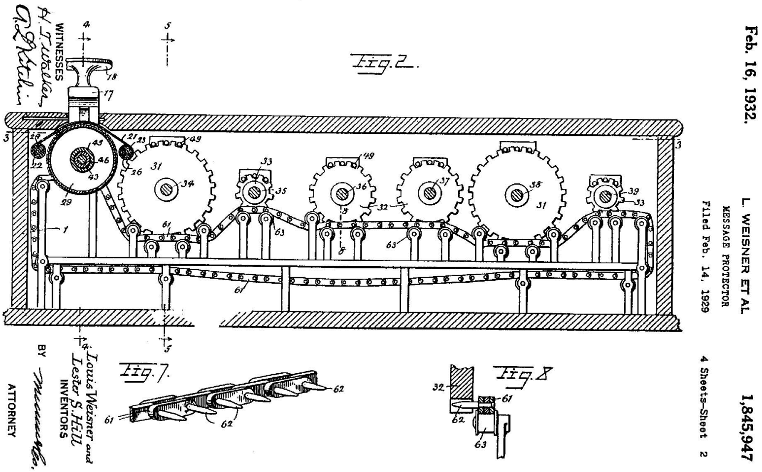
d) none of the above



https://xkcd.com/184/

Mechanical Message Protector

- Louis Weisner & Lester Hill
- U.S. patent number 1845947 from 1929



Hill Cipher

- linear transformation of the digitized alphabet
- A.[uncoded message] $=$ [coded message]

$$A \left[\begin{array}{ccc} \text{col1=start of message} & \ldots & \text{coln=end of message} \end{array} \right]$$

- mechanical device: 6×6 matrix
- method is vulnerable to those that intercept enough vector correspondences (uncoded and coded) because of its linearity.
- final project idea: linear methods of cracking the cipher
- more powerful computers - RSA cryptosystem
- Maple

For the Hill cipher

a) $A_{n\times n}$ [original message]$_{n\times p}$ = [coded message]$_{n\times p}$
b) to decode, we apply an invertible matrix to the coded message and read the message along the rows
c) the method is vulnerable to those that intercept enough coded/decoded vector correspondences because of its linearity
d) all of the above
e) two of the above

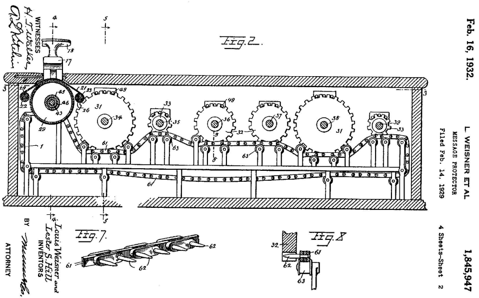


Image Credit: 1932 Patent Application 1,845,947 https://patents.google.com/patent/US1845947A/en