2.8 Spaces

SUBSPACE, THE [0, cv & u+v] FRONTIER

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Subspace

nonempty set of vectors on which we can do + and scalar mult:

- $\vec{v}_1 + \vec{v}_2$ is in the space whenever the individual vectors are
- $c\vec{v}$ is in the space whenever $\vec{v}$ is and c is a real scalar

$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ versus $t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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- line through $\vec{0}$
- $\mathbb{R}^2$

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- $\mathbb{R}^3$

Basis for a Subspace

- a linearly independent spanning set

  like $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ but not $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

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  $\begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \textcircled{1} & 4 & 7 & 0 \\ 0 & \textcircled{-3} & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ missing a pivot for z

- span of a set of vectors is a subspace
- dimension

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Difference Between a Basis and the Subspace Itself

- subspace is generated by the basis via all linear combinations, i.e. the span

basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ subspace: $s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

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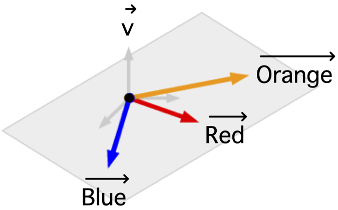
- exclusive club $xy0$-plane in $\mathbb{R}^3$



basis?

Two Special Subspaces Related to a Matrix

- column space of A: span of the columns of A
- null space of A : set of solutions to $A\vec{x} = \vec{0}$

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$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{bmatrix}$ pivot columns basis of the column space

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{r_2' = -r_1 + r_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

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geometry of entire column space?

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null space write $A\vec{x} = \vec{0}$ solutions by parameterizing

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back substitution $x_2 = s$, $x_4 = t$, row 2: [0 0 -1 -3 0] $x_3 = -3t$,

row 1: $x_1 = -2x_2 - 3x_3 - 4x_4 = -2s - 3(-3t) - 4t = -2s + 5t$

$$s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \text{ basis: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\} \text{ geo?}$$

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Two Special Subspaces Related to Another Matrix

- column space of A: span of the columns of A
- null space of A: set of solutions to $A\vec{x} = \vec{0}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) What is a basis for the column space of A?
b) What is the column space of A?
c) What is the geometry of the column space of A?

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$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) What is a basis for the column space? $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

b) What is the column space of A? $s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

c) What is the geometry of the column space of A?
infinite volume in $\mathbb{R}^3$ is all of $\mathbb{R}^3$

Two Special Subspaces Related to Another Matrix

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$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) What is the nullspace of A?

e) What is a basis for the nullspace of A?

f) What is the geometry of the nullspace of A?

Two Special Subspaces Related to Another Matrix

- column space of A: span of the columns of A
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d) What is the nullspace of A? Augment with 0 and

parameterize $[A0] = \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 & 0 \\ 0 & \textcircled{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{bmatrix}$

x_3 is free so $x_3 = t$.

row 3: [0 0 0 1 0] $x_4 = 0$

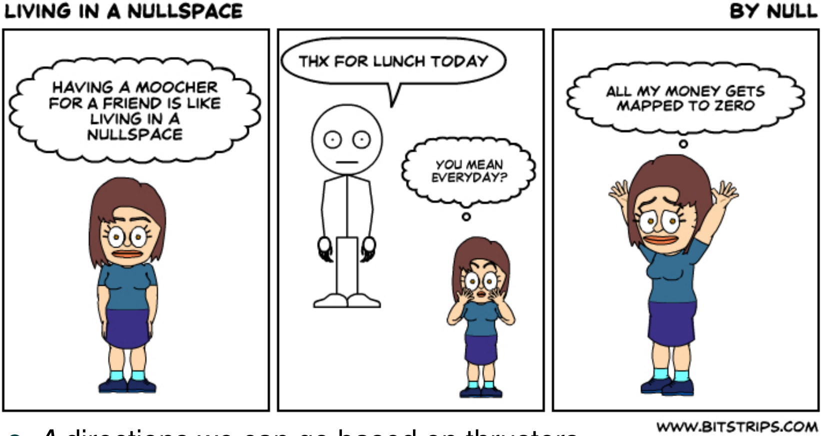
row 2: [0 1 1 0 0] $x_2 + x_3 = 0$ so $x_2 = -x_3 = -t$

row 1: [1 0 0 1 0] $x_1 + x_4 = 0$ so $x_1 = -x_4 = -0 = 0$

nullspace is $\begin{bmatrix} 0 \\ -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ e) basis is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

f) What is the geometry of the nullspace of A? line in $\mathbb{R}^4$

Applications



- *A* directions we can go based on thrusters
- *A* rate of return on investments
- *A* room illumination
- *A* set of map directions (vectors) at entrance to forest

Null Space and Column Space

Which of the following statements about are true about the nullspace (or null space) and column space of $M = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Note that M is row equivalent to $\begin{bmatrix} 1 & 4 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$ and when M is augmented with a generic vector and reduced to Gaussian, the last row becomes $\begin{bmatrix} 0 & 0 & b_1 - 2b_2 + b_3 \end{bmatrix}$

a) the column space is the plane $b_1 - 2b_2 + b_3 = 0$ in $\mathbb{R}^3$

b) the column space is the plane $s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ in $\mathbb{R}^3$

c) the nullspace is $\vec{0}$ in $\mathbb{R}^2$

d) all of the above

e) two of the above

Maple's Null Space and Column Space

Using with(LinearAlgebra): with(plots):
M:=Matrix([[1,4],[2,5],[3,6]]); ColumnSpace(M); Maple outputs
the equivalent of $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. Are these vectors even in the column space? We can check $b_1 - 2b_2 + b_3 = 0$

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For $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $b_1 - 2b_2 + b_3 = 1 - 2(0) - 1 = 0$

For $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $b_1 - 2b_2 + b_3 = 0 - 2(1) + 2 = 0$

basis with 1s and 0s in the first coordinates. Spans the same space, the same plane in $\mathbb{R}^3$. It's a basis, not entire space.

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NullSpace(M); outputs $\emptyset$ as no basis for entire nullspace $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Null Space of a Non-Square Matrix

The null space of a non-square matrix is a subspace of

a) $\mathbb{R}^{\text{number of rows}}$

b) $\mathbb{R}^{\text{number of columns}}$

c) further work must be done to tell

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I Famili, BO Palsson - Biophysical journal, 2003 - Elsevier
… between the reaction rate vectors, v, and time derivative of metabolite concentrations, dx/dt or x ˙ . Each two subspaces in the domain (ie, the **null space** and row space) and codomain (ie, the left **null space** and **column space**) form orthogonal pairs with one another …
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