2.9 Dimension and Rank-Nullity Theorem

SUBSPACE, THE [0, cv & u+v] FRONTIER

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SUBSPACE, THE [0, cv & u+v] FRONTIER

BASIS, l.i. + span, efficient represent

COLUMN SPACE, SPAN OF THE COLUMNS

NULL SPACE, SOLUTIONS TO Ax=0

Basis Review

- A basis for a subspace is a set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ that is a spanning set for the subspace and is linearly independent
- Any vector can be uniquely written as a linear combination of the vectors in the basis

Example: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

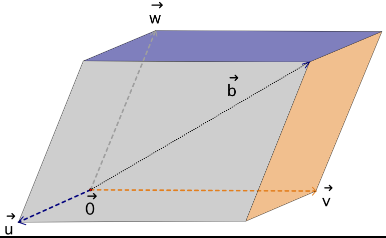
$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 3 & -6 & -5 & b_2 \\ 2 & -1 & 1 & b_3 \end{bmatrix} \xrightarrow{\text{Gaussian}} \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 3 & 3 & -2b_1 + b_3 \\ 0 & 0 & -2 & -3b_1 + b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -5 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{Gaussian}} \begin{bmatrix} \boxed{1} & -2 & -1 & 0 \\ 0 & \boxed{3} & 3 & 0 \\ 0 & 0 & \boxed{-2} & 0 \end{bmatrix}$$

What do Bases have in Common?

- $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also a basis for $\mathbb{R}^3$

Every basis has exactly the same number of vectors. The number is the dimension.



Change of Basis Coordinates

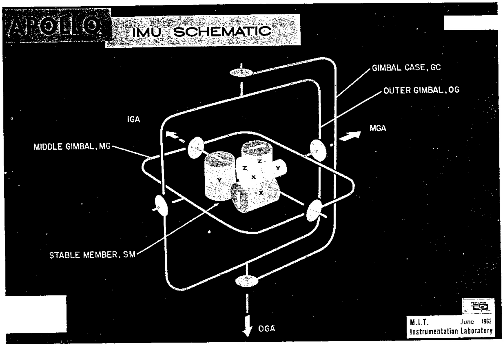


Figure 1

U.S. Gov't

`https://www.hq.nasa.gov/alsj/imu-2.jpg`

Rank and Nullity

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rank of A is the dimension of the column space

nullity of A is the dimension of the null space

Rank and Nullity Example

What is the rank and nullity of $A = \begin{bmatrix} 3 & -1 & 7 & -6 \\ 4 & -1 & 9 & -7 \\ -2 & 1 & -5 & 5 \end{bmatrix}$?

$$\begin{bmatrix} 3 & -1 & 7 & -6 \\ 4 & -1 & 9 & -7 \\ -2 & 1 & -5 & 5 \end{bmatrix} \xrightarrow[r_3'=\frac{2}{3}r_1+r_2]{r_2'=-\frac{4}{3}r_1+r_2} \begin{bmatrix} 3 & -1 & 7 & -6 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 1 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}$$

$$\xrightarrow{r_3'=-r_2+r_3}$$

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$$\xrightarrow{r_3'=-r_2+r_3}$$

$$\begin{bmatrix} \boxed{3} & \boxed{-1} & 7 & -6 \\ \boxed{4} & \boxed{-1} & 9 & -7 \\ \boxed{-2} & \boxed{1} & -5 & 5 \end{bmatrix} \xrightarrow{\textit{Gaussian}} \begin{bmatrix} \boxed{3} & -1 & 7 & -6 \\ 0 & \boxed{\frac{1}{3}} & -\frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for column space $\left\{ \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ rank=2

Rank and Nullity Example

$$\left[\begin{array}{rrrrr} 3 & -1 & 7 & -6 & 0 \\ 4 & -1 & 9 & -7 & 0 \\ -2 & 1 & -5 & 5 & 0 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \end{array} \\ \left[\begin{array}{rrrrr} 3 & -1 & 7 & -6 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Rank and Nullity Example

$$\begin{bmatrix} 3 & -1 & 7 & -6 & 0 \\ 4 & -1 & 9 & -7 & 0 \\ -2 & 1 & -5 & 5 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & \boxed{x_3} & \boxed{x_4} & = \end{array} \\ \begin{bmatrix} 3 & -1 & 7 & -6 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x_3 = s$ free variable

$x_4 = t$ free variable

row 2: $\frac{1}{3}x_2 - \frac{1}{3}x_3 + 1x_4 = 0$ so $\frac{1}{3}x_2 = \frac{1}{3}s - 1t$, $x_2 = s - 3t$

Rank and Nullity Example

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row 1: $3x_1 = x_2 - 7x_3 + 6x_4 = s - 3t - 7s + 6t$

$= -6s + 3t$ so $x_1 = -2s + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \text{ basis } \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\text{nullity}(A) = 2$

Rank-Nullity Theorem

If a matrix A has n columns then $rank(A) + nullity(A) = n$

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Suppose A is a 3×5 matrix with nullity 4. Give a geometric description of the column space of A.

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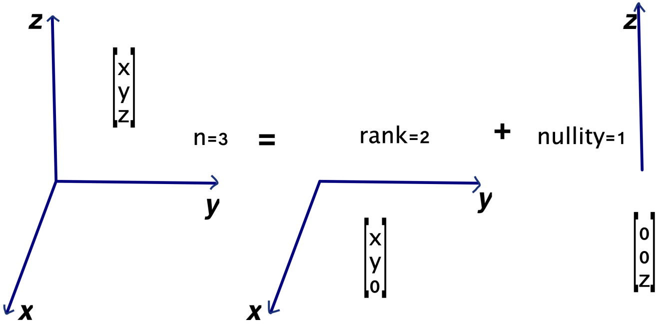
Suppose A is a 3×5 matrix with nullity 4. Give a geometric description of the column space of A.

- $A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$ The column space is a subspace of $\mathbb{R}^3$.
- $rank(A) + \underbrace{4}_{nullity} = \underbrace{5}_{\#\ columns}$
- $rank(A) = 5 - 4 = 1$
- The column space of A is a 1 dimensional subspace of $\mathbb{R}^3$, so it is a line through the origin.

Rank-Nullity Theorem Visualized for Projection

Rank is the number of vectors that are left as basis vectors and the nullity is the number that are squashed.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. TFAE

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix
- A has n pivot positions
- The equation $A\vec{x} = \vec{0}$ has only the trivial solution
- The columns of A form a linearly independent set
- The equation $A\vec{x} = \vec{b}$ has a unique solution for each $\vec{b}$ in $\mathbb{R}^n$
- The columns of A span $\mathbb{R}^n$
- A^T is invertible
- The column space of A is $\mathbb{R}^n$
- $\text{rank}(A) = n$
- The null space of A is $\{\vec{0}\}$
- $\text{nullity}(A) = 0$

Apply the Rank-Nullity Theorem

Apply the rank-nullity theorem to $\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -5 & 1 & 0 \\ -1 & 1 & -1 & 8 \end{bmatrix}$

a) rank + nullity = 3 + 0 = 3

b) rank + nullity = 2 + 1 = 3

c) rank + nullity = 3 + 1 = 4

d) rank + nullity = 2 + 2 = 4

e) other

Apply the Rank-Nullity Theorem

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -5 & 1 & 0 \\ -1 & 1 & -1 & 8 \end{bmatrix} \xrightarrow[r_3'=r_1+r_3]{r_2'=-3r_1+r_2} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & -1 & -1 & 12 \end{bmatrix}$$

$$\xrightarrow{r_3'=r_2+r_3} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply the Rank-Nullity Theorem

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -5 & 1 & 0 \\ -1 & 1 & -1 & 8 \end{bmatrix} \xrightarrow[r_3'=r_1+r_3]{r_2'=-3r_1+r_2} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & -1 & -1 & 12 \end{bmatrix}$$

$$\xrightarrow{r_3'=r_2+r_3} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \right\}$ so rank $= 2$

Apply the Rank-Nullity Theorem

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -5 & 1 & 0 \\ -1 & 1 & -1 & 8 \end{bmatrix} \xrightarrow{\text{Gaussian}} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \right\}$ so rank $= 2$

$$\begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 3 & -5 & 1 & 0 & 0 \\ -1 & 1 & -1 & 8 & 0 \end{bmatrix} \xrightarrow{\text{Gaussian}} \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 1 & 1 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nullspace $A\vec{x} = 0$ is

$x_3 = s, x_4 = t, x_2 = -s + 12t, x_1 = 2(-s + 12t) - 4t = -2s + 20t$

$$\begin{bmatrix} -2s + 20t \\ -s + 12t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 20 \\ 12 \\ 0 \\ 1 \end{bmatrix} \text{ with basis } \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ 12 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and nullity 2 Maple

Don't Make this Column Space Mistake!

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & -5 & 1 & 0 \\ -1 & 1 & -1 & 8 \end{bmatrix} \xrightarrow{\text{Gaussian}} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

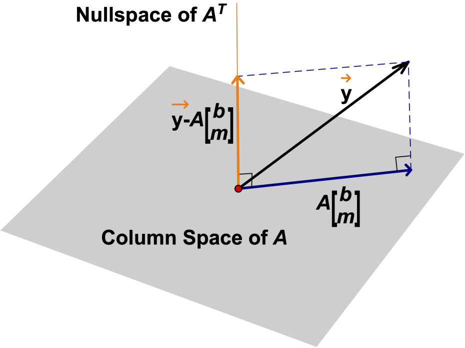
basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ or others

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is NOT a basis for the column space

vectors in the column space satisfy $-2b_1 + b_2 + b_3 = 0$ from the [0 0 0 0 b3-2b1+b2] row

none of the reduced matrix vectors satisfy this, like column 4: $-2b_1 + b_2 + b_3 = -2(4) + (-12) + (0) \neq 0$ but all the original matrix columns plus Maple's column space basis vectors do.

Least Squares Linear Regression



Let $\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} y\text{-intercept} \\ \text{slope} \end{bmatrix}$

apply nullspace def: $A^T(\vec{y} - A\begin{bmatrix} b \\ m \end{bmatrix}) = \vec{0}$ so $A^T\vec{y} = (A^TA)\begin{bmatrix} b \\ m \end{bmatrix}$

Also, if (A^TA) is invertible then $\begin{bmatrix} b \\ m \end{bmatrix} = (A^TA)^{-1}A^T\vec{y}$ lets us solve for the best fit line!

Error-correcting Codes

A generating matrix for a linear code, G, takes in k-bit string of information, $\vec{x}$, and outputs an n-bit error-protected blocks by applying the transformation $T(\vec{x}) = G\vec{x}$.

The column-space of G is the set of all codewords

A check matrix for a linear code is a matrix H with the property that $H\vec{z} = \vec{0}$ if and only if $\vec{z}$ is a codeword.

The null space of H is the set of all codewords

https://phys.org/news/2012-02-error-correcting-codes-fastest-transmission.html

http://www.math.toronto.edu/afenyes/writing/error-correction%20(october%202015).pdf