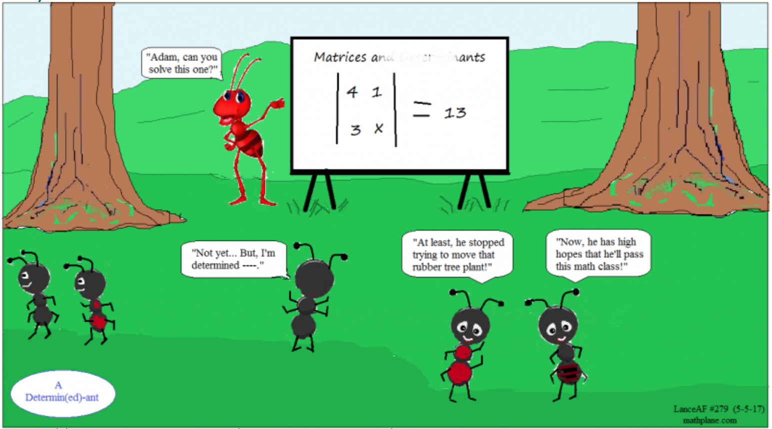
3.1, 3.2, and 3.3 Determinants

a) invertibility of a 2×2 matrix
b) determinant 1 (or -1) coding matrix with integer entries will ensure we don't pick up fractions in the decoding matrix
c) both of the above



http://www.mathplane.com/gate_dwebcomics/math_comics_archive_spring_2017

$2 \times 2, 3 \times 3$ and 4×4 *Determinants*

- Maple

$2 \times 2, 3 \times 3$ and 4×4 *Determinants*

- Maple
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{\text{first 2 columns}} \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$

  3 main diagonals: $a \cdot e \cdot i + b \cdot f \cdot g + c \cdot d \cdot h$

  minus 3 off diagonals: $-c \cdot e \cdot g - a \cdot f \cdot h - b \cdot d \cdot i$

$2 \times 2, 3 \times 3$ and 4×4 *Determinants*

- Maple
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  minus 3 off diagonals: $-c \cdot e \cdot g - a \cdot f \cdot h - b \cdot d \cdot i$
- Determinant terms
  2×2 has 2 terms, 3×3 has 6 terms, 4×4 has 24 terms.
  Do you see a pattern?

1683 Takakazu Shinsuke Seki computed
$2 \times 2, 3 \times 3, 4 \times 4$ and 5×5 determinants

Cofactor or Laplace Expansion of Determinant

$$\sum_{1}^{n} a_{ij}(-1)^{i+j}|\text{matrix obtained by eliminating row } i \text{ and column } j|$$

where we have fixed i or j to expand along

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \sum_{1}^{n} a_{2j}C_{2j} = \sum_{1}^{n} a_{2j}(-1)^{2+j} \text{ Minor}_{2j}$$

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$$\begin{vmatrix} \boxed{1} & 4 & 7 \\ \boxed{2} & \boxed{5} & \boxed{8} \\ \boxed{3} & 6 & 9 \end{vmatrix} \qquad \begin{vmatrix} 1 & \boxed{4} & 7 \\ \boxed{2} & \boxed{5} & \boxed{8} \\ 3 & \boxed{6} & 9 \end{vmatrix} \qquad \begin{vmatrix} 1 & 4 & \boxed{7} \\ \boxed{2} & \boxed{5} & \boxed{8} \\ 3 & 6 & \boxed{9} \end{vmatrix}$$

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$$2(-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} + 5(-1)^{2+2}\begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} + 8(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix}$$

$$2(-1)(4 \cdot 9 - 6 \cdot 7) + 5(1)(1 \cdot 9 - 3 \cdot 7) + 8(-1)(1 \cdot 5 - 3 \cdot 4)$$

$$= 12 - 60 + 48 = 0$$

1772 orbits of the inner planets

Row 1 vs Column 1 Cofactor/Laplace Expansion

$$\sum_{1}^{n} a_{ij}(-1)^{i+j}|\text{matrix obtained by eliminating row } i \text{ and column } j|$$

column 1 $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

$$=1(-1)^{1+1}\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} + 3(-1)^{3+1}\begin{vmatrix} 4 & 7 \\ 5 & 8 \end{vmatrix} = 0$$

$$=1(1)(5 \cdot 9 - 6 \cdot 8) + 2(-1)(4 \cdot 9 - 6 \cdot 7) + 3(1)(4 \cdot 8 - 5 \cdot 7)$$

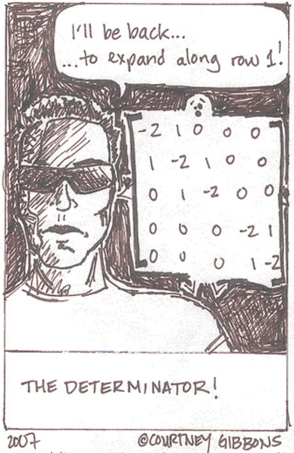
$= -3 + 12 - 9 = 0 =$ determinant or det

row 1 $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

$$=1(-1)^{1+1}\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} + 4(-1)^{1+2}\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} + 7(-1)^{1+3}\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$$

$$=1(1)(5 \cdot 9 - 6 \cdot 8) + 4(-1)(2 \cdot 9 - 3 \cdot 8) + 7(1)(2 \cdot 6 - 3 \cdot 5)$$

$= -3 + 24 - 21 = 0 =$ determinant or det



http://brownsharpie.courtneygibbons.org/comic/determinator/

$\sum_{1}^{n} a_{1j} \cdot (-1)^{1+j}$ · Det of matrix obtained by eliminating row 1 and column j where $i = 1$ is fixed, $j = 1..5$

$$-2 \cdot (-1)^{1+1} \begin{vmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{vmatrix} + 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{vmatrix} + 0s$$

Taking Advantage of 0s

$$\begin{vmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

By hand, compute via the cofactor/Laplace expansion:

Step 1: down the 1st **column** to take advantage of the 0s.

Step 2: down the 1st **column** of the resulting 4×4 matrix

Step 3: along the 3rd **row** of the resulting 3×3 matrix:

$$5(-1)^{1+1} \begin{vmatrix} 1 & 4 & 3 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 5(1) \cdot 1(-1)^{1+1} \begin{vmatrix} 2 & 6 & 3 \\ 3 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 5(1) \cdot 1(1) \cdot 2(-1)^{3+3} \begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix}$$

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$$= 5(1) \cdot 1(1) \cdot 2(-1)^{3+3} \begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix}$$

$$= 5 \cdot 1 \cdot 2(2 \cdot 4 - 3 \cdot 6) = -100$$

Algebraic Properties of Determinant

1812 Cauchy explored determinants, minors and cofactors and proved $|AB| = |A||B|$

Compute the matrices and their determinants.

a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ vs. det of matrix after $r_2' = -3r_1 + r_2$: $\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}$

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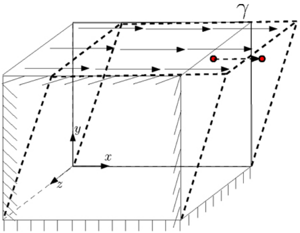
d) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ vs. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}^T = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$



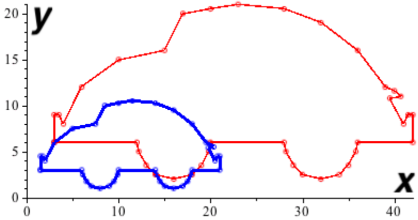


Image 1: Modeling of Hot-Mix Asphalt Compaction: A Thermodynamics-Based Compressible Viscoelastic Model

[FHWA-HRT-10-065], rest of images made using VLA program by Herman and Pepe *Visual Linear Algebra*

- $r_j' = kr_i + r_j$ shear $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.object same determinant
- $r_i \leftrightarrow r_j$ reflect $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.object negative determinant
- $r_j' = cr_j$ scale $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$.object scales determinant
- transpose preserves determinant

Determinant of Triangular Matrix and Inverse

A triangular matrix has 0s below the diagonal (such as in Gaussian to row echelon form), or 0s above the diagonal:

- $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix}$ or $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{vmatrix}$
  By-hand, what is the determinant of a triangular matrix?

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  By-hand, what is the determinant of a triangular matrix?

- $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1\cdot 4 - 3\cdot 2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$
  What is its determinant of the inverse
  and how does it compare to the original?

Determinant of Invertible Matrices

- Assume A is invertible. Then
  $AA^{-1} = I$
  $|AA^{-1}| = |I|$
  $|A||A^{-1}| = 1$
  so $|A| \neq 0$ because $0 \cdot |A^{-1}| \neq 1$ and $|A^{-1}| = \frac{1}{|A|}$

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- $A \rightarrow I$
  elementary row operations
  don't change a determinant
  from 0 to nonzero
  since $|I| = 1$ then $|A| \neq 0$

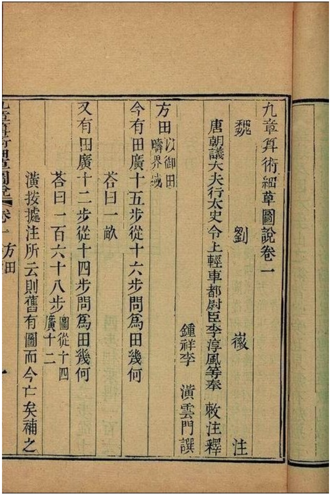
Invertible Matrix Theorem for $A_{n\times n}$

The following are equivalent (TFAE):

- A is an invertible matrix
- A is row equivalent to the $n \times n$ identity matrix
- A has n pivot positions
- $A\vec{x} = \vec{0}$ has only the trivial solution
- columns of A form a linearly independent set
- $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$
- columns of A span $\mathbb{R}^n$
- there is an $n \times n$ matrix C such that $CA = I$
- there is an $n \times n$ matrix D such that $AD = I$
- A^T is an invertible matrix
- $|A| \neq 0$

Determinant 0 Matrices

Suppose the determinant of matrix A is zero. How many solutions does the system $A\vec{x} = 0$ have?


九章筭術細草圖說卷一
魏 劉 徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 敕注釋
鍾祥李 潢雲門譔
方田以御田疇界域
今有田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
荅曰一百六十八步圖從十四廣十二
潢按據注所云則舊有圖而今亡矣補之

The Nine Chapters on the Mathematical Art

Trivial Solution

We find that for a square coefficient matrix A,
the homogeneous system $A\vec{x} = \vec{0}$,
has only the trivial solution $\vec{x} = \vec{0}$.
This means that

a) A has a 0 determinant
b) A has a nonzero determinant
c) This tells us nothing about the determinant

A short survey of some recent **applications** of **determinants**
PR Vein - Linear Algebra and its **Applications**, 1982 - Elsevier
Determinants declined in prestige from the mid-nineteenth century onwards and are now best known for their **applications** in matrix theory, where they appear in a subsidiary role. However, during the last thirty years **determinants** have arisen independently of matrices in …
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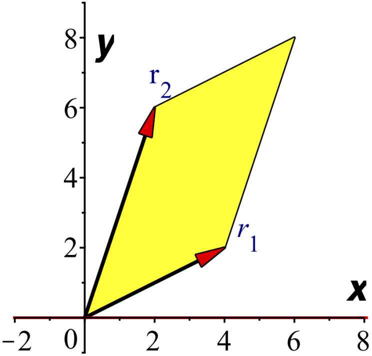
[BOOK] **Determinants** and their **applications** in mathematical physics
R Vein, P Dale - 2006 - books.google.com
The last treatise on the theory of **determinants**, by T. Muir, revised and enlarged by WH Metzler, was published by Dover Publications Inc. in 1960. It is an unabridged and corrected republication of the edition ori-nally published by Longman, Green and Co. in 1933 and …
Cited by 198 Related articles All 16 versions

Google Scholar search of applications of determinants

Geometric Properties of Determinant 2×2

$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$ $\det A = \det A^T$ so examine the rows and the unit span



$tr_1 + sr_2, 0 \leq s, t \leq 1$

image made using VLA program by Herman and Pepe *Visual Linear Algebra*

strict Gaussian $r_2' = -\frac{1}{2} r_1 + r_2$ or equivalently the shear

$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 5 \end{bmatrix}$ preserves determinant

Geometric Properties of Determinant 2×2

In general $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ or $r_2' = tr_1 + r_2$ takes the second row to a vector that ends on the line parallel to _ through the tip of _

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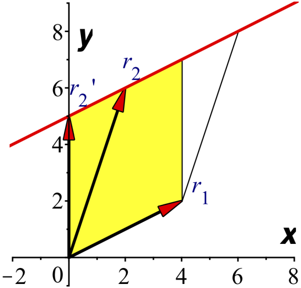
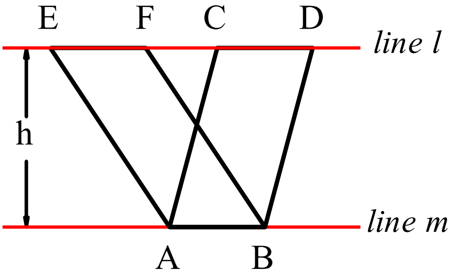


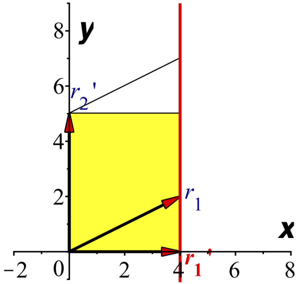
image made using VLA program by Herman and Pepe *Visual Linear Algebra*

we are acting on the rows rather than the columns

it isn't visualized as a vertical shear—it is an r_1 shear parallel to r_1 through the tip of r_2

Geometric Properties of Determinant 2×2





images made using VLA program by Herman and Pepe *Visual Linear Algebra*

strict Gauss-Jordan $r_1' = -\frac{2}{5}r_2 + r_1$ or equivalently

$$\begin{bmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$ determinant = area = 20

Strict replacements shears unit span parallelograms to rectangles with the same area. We may have had to swap rows, changing only the sign of the determinant.

$|\text{determinant}|$ = area parallelogram

Geometric Properties of Determinant 3×3

$|\text{determinant}|$ = ____ for 3 column vectors in a 3×3 matrix

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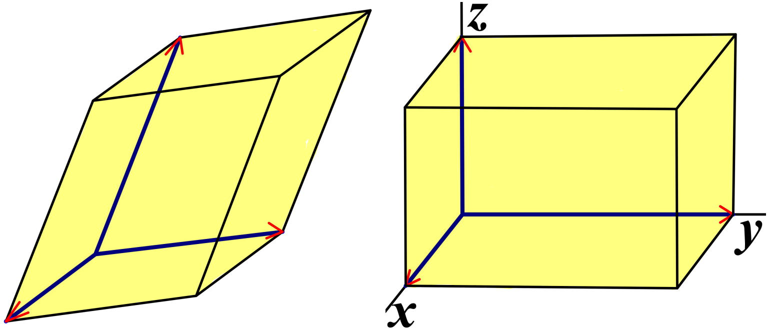
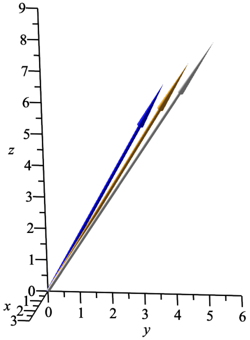
$|\text{determinant}|$ = ____ for 3 column vectors in a 3×3 matrix

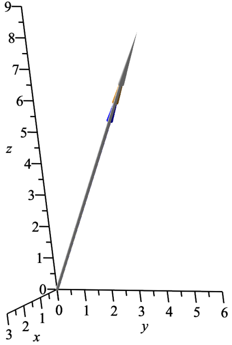


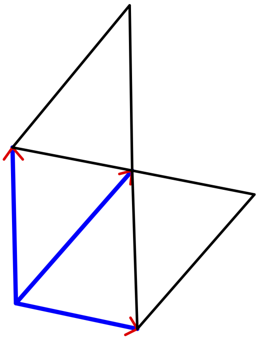
volume of unit span parallelepiped

1773 Joseph-Louis Lagrange

Geometric Interpretation of 0 Determinant







0 determinant? degenerate figure—smushed

3 vectors all in the same plane giving 0 volume for 3×3

2 vectors on same line giving 0 area for 2×2

Row Equivalent Rectangle?

The area of the parallelogram formed by considering the vectors in $A = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$ is $|A| = 8$. Can we find a rectangle that creates a matrix that is row equivalent to A with the same area?

a) impossible with the conditions given
b) yes

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$r_1' = -\frac{5}{8}6r_2 + r_1$ or $\begin{bmatrix} 1 & -\frac{5}{8}6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 0 & \frac{8}{5} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & \frac{8}{5} \end{bmatrix}$