5.1 and 5.2 Eigenvalues and Eigenvectors

- If $A\vec{x}$ realigns on the same line as $\vec{x}$ via $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an *eigenvector* and λ is an *eigenvalue*
- $A\vec{x} = \lambda\vec{x}$ matrix multiplication to scalar multiplication by λ

Eigenvalues and Eigenvectors of a Horizontal Shear

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Vectors on the x-axis are fixed in the animation, i.e. $\lambda = 1$

Try it: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} =$

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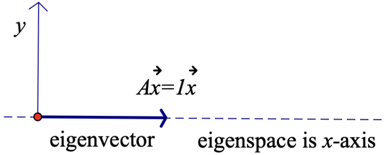
Try it: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = 1 \begin{bmatrix} x \\ 0 \end{bmatrix}$ so $A\vec{x} = 1\vec{x}$

So anything on the x-axis, like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 1. The eigenspace for $\lambda = 1$ is the entire set of eigenvectors corresponding to this eigenvalue, the $x-axis$.



Eigenvalues & Eigenvectors of Reflection across $y = x$

Consider what else realigns on the same line through the origin.

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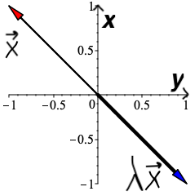

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -x \\ x \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = -1 \begin{bmatrix} -x \\ x \end{bmatrix}$$

Maple $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ shows orthogonal eigenspaces:

$\lambda = 1$ has $y = x$ eigenspace with Maple basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$ has $y = -x$ eigenspace with Maple basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

π − *Rotation about z-axis in* $\mathbb{R}^3$

Consider what realigns on the same line through the origin.

$\pi -$ *Rotation about z-axis in* $\mathbb{R}^3$

Consider what realigns on the same line through the origin.
any vector on the $0-0-z$ line has $\lambda = 1$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ 0 \end{bmatrix}$$ $\lambda = -1$ eigenspace is plane

Maple

The eigenspace corresponding to the eigenvalue $\lambda = -1$ is given by span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, the plane, and the eigenspace corresponding to the $\lambda = 1$ eigenspace is given by span $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, the z-axis.

Eigenvalues and Eigenvectors Algebraically

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$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix}$

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multiply out and solve for λ:

$0 = \frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4} = \lambda^2 - \lambda = \lambda(\lambda - 1)$

So $\lambda = 0$ and $\lambda = 1$ are eigenvalues

Eigenvalues and Eigenvectors Algebraically

eigenvectors of A are in the nullspace of $(A - \lambda I)$. We want non-trivial solutions, so we obtained the eigenvalues from the characteristic equation determinant $(A - \lambda I) = 0$.

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$$\lambda = 0 \begin{bmatrix} \frac{1}{2} - 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} - 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

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$y = t$ then backsub into row 1: $\frac{1}{2}x + \frac{1}{2}y = 0$ so $x = -t$. Then nullspace of $A - 0I$ is $\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, i.e. $y = -x$ line

$\lambda = 1$

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$$\lambda = 1 \begin{bmatrix} \frac{1}{2} - 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} - 1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \xrightarrow{r_2' = r_1 + r_2} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y = t, x = t$ so nullspace of $A - I$ is $t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e. $y = x$ line

Eigenvalues Algebraically By the Quadratic Formula

$$A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix} \; A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix}$$

characteristic equation:

0=det$(A - \lambda I) = (3-\lambda)(-1-\lambda) - (-2)(1) = \lambda^2 - 2\lambda - 1$

solving for eigenvalues of $a\lambda^2 + b\lambda + c$:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{--2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = 1 \pm \sqrt{2}$$

so $\lambda = 1 + \sqrt{2} \approx 2.414$ and $\lambda = 1 - \sqrt{2} \approx -.414$

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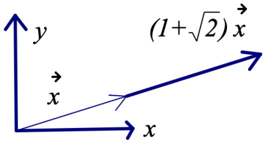
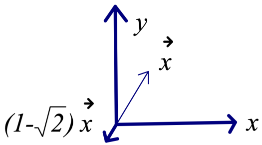
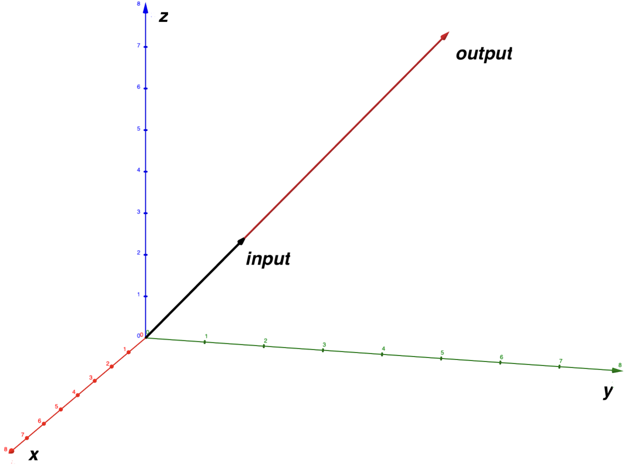




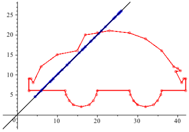
Implications of the Algebra and Geometry

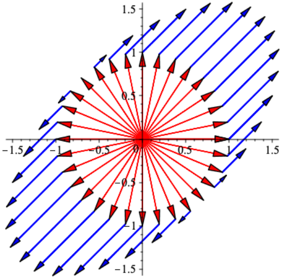
It is always the case that scaling a vector by λ is the same as changing its length by λ. Why? The length of $\lambda\vec{x}$ is $\sqrt{\lambda\vec{x}\cdot\lambda\vec{x}}=\sqrt{\lambda^2\vec{x}\cdot\vec{x}} = \lambda\sqrt{x\cdot x}$



Implications of the Algebra and Geometry

- $\lambda = 0$ is an eigenvalue of A if a line (or more) gets smushed to the origin i.e. $A\vec{x} = 0\vec{x} = \vec{0}$ has a non-trivial solution and A is not invertible





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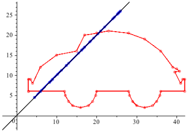

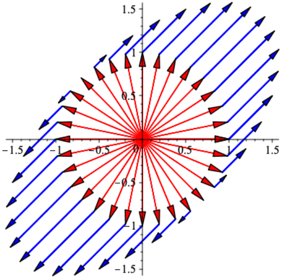

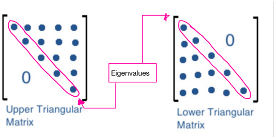

http://mathonline.wikidot.com/triangular-matrices

- The eigenvalues of a triangular matrix: $0 = \text{determinant } (A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)...(a_{nn} - \lambda)$ are exactly the diagonal entries of that triangular matrix

Another Example

Solve for the eigenvalues of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: $|(A - \lambda I)| = 0$

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0= $\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = \lambda^2 + 1 \qquad \lambda = \dfrac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)}$

What geometric transformation is this? Consider why nothing (aside from $\vec{0}$) realigns on the same line through the origin.

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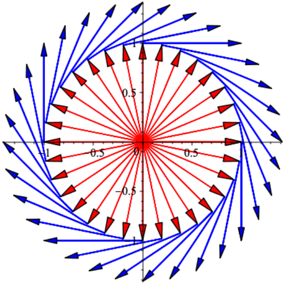
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix}$$



Probability, Markov, or Stochastic Matrix

A basketball team has a 60% probability of winning their next game if they have won their previous game but only a 30% probability of winning their next game if they have lost their previous game. Let $\vec{x}_k = \begin{bmatrix} \text{\% chance of winning game } k \\ \text{\% chance of losing game } k \end{bmatrix}$.

Then $\vec{x}_{k+1} = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} \vec{x}_k$

What are the eigenvalues?

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$= \lambda^2 - 1.3\lambda + .3 = (\lambda - 1)(\lambda - .3)$

The eigenspace corresponding to the larger magnitude eigenvalue is especially useful! Here $\lambda = 1$ dominant

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$\begin{bmatrix} .6-1 & .3 & 0 \\ .4 & .7-1 & 0 \end{bmatrix} \xrightarrow{r_2' = r_1 + r_2} \begin{bmatrix} -.4 & .3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad t \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$ i.e. $y = \frac{4}{3}x$

$\lambda = 1$ *in Probability, Markov, or Stochastic Matrix*

Here $\lambda = 1$ is especially useful since that leads to a steady state in the long run (we'll see more about why in 5.6):

$A\vec{x} = 1\vec{x} = \vec{x}$ for $\vec{x} = t \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$ in the eigenspace for $\lambda = 1$

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Thus in the long run we stabilize to $\begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} \sim \begin{bmatrix} .43 \\ .57 \end{bmatrix}$

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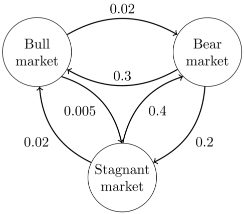

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SIAM Review, 2006, Vol. 48, No. 3 : pp. 569-581

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Google's success derives in large part from its PageRank algorithm, which ranks the