5.6 Discrete Dynamical Systems

Yearly changes in populations in a predator-prey model:

$$\begin{aligned} \text{Fox}_k &= .6\text{Fox}_{k-1} + .4\text{Rabbit}_{k-1} \\ \text{Rabbit}_k &= -.125\text{Fox}_{k-1} + 1.2\text{Rabbit}_{k-1} \end{aligned}$$

- Write the system as a matrix vector equation with foxes as the x-value and rabbits as the y-value.

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- If we know an initial condition $\vec{x}_0 = \begin{bmatrix} \text{Fox}_0 \\ \text{Rabbit}_0 \end{bmatrix}$ then we know the populations at discrete times $A^k\vec{x}_0$.
- Long-term behavior?

Long-term Behavior via Eigenvalues and Eigenvectors

If eigenvectors of $A_{n\times n}$ form a basis for $\mathbb{R}^n$, call them $\vec{x}_1, ...\vec{x}_n$, then any initial condition is in the span:
$\vec{x}_0 = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$

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Now we apply A and matrix algebra:

$\vec{x}_1 = A\vec{x}_0$
$= A(c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_n)$
$= A(c_1\vec{x}_1) + A(c_2\vec{x}_2) + \cdots + A(c_n\vec{x}_n)$
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$= c_1(\lambda_1\vec{x}_1) + c_2(\lambda_2\vec{x}_2) + \cdots + c_n(\lambda_n\vec{x}_n)$

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&= c_1(A\vec{x}_1) + c_2(A\vec{x}_2) + \cdots + c_n(A\vec{x}_n)\\
&= c_1(\lambda_1\vec{x}_1) + c_2(\lambda_2\vec{x}_2) + \cdots + c_n(\lambda_n\vec{x}_n)
\end{aligned}$$

Keep turning matrix multiplication to scalar multiplication!

$$\begin{aligned}
\vec{x}_k &= A^k\vec{x}_0\\
&= c_1\lambda_1^k\vec{x}_1 + c_2\lambda_2^k\vec{x}_2 + \cdots + c_n\lambda_n^k\vec{x}_n
\end{aligned}$$

eigenvector decomposition

Long-term Behavior of Foxes and Rabbits

$$\begin{bmatrix} \text{Fox}_k \\ \\ \text{Rabbit}_k \end{bmatrix} = \begin{bmatrix} \frac{6}{10} & \frac{4}{10} \\ \\ -\frac{125}{1000} & \frac{12}{10} \end{bmatrix} \begin{bmatrix} \text{Fox}_{k-1} \\ \\ \text{Rabbit}_{k-1} \end{bmatrix}$$

- If the eigenvectors of A form a basis for $\mathbb{R}^2$ then $\vec{x}_k = A^k \vec{x}_0 = c_1 \lambda_1^k \vec{x}_1 + c_2 \lambda_2^k \vec{x}_2$
- Maple: eigenvectors, span for eigenvector decomposition

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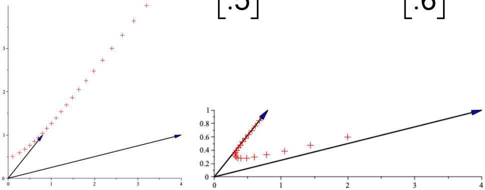
- Maple: eigenvectors, span for eigenvector decomposition $\vec{x}_k = c_1 (\frac{11}{10})^k \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix} + c_2 (\frac{7}{10})^k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- Trajectory diagrams for $\vec{x}_0 = \begin{bmatrix} .1 \\ .5 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 2 \\ .6 \end{bmatrix}$



Never cross me!

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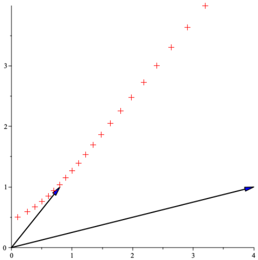
Limits Applied to Diverse Objects

- Calculus with Analytic Geometry:
  - limits used in defining derivatives and integrals
  - $\lim\limits_{b\to\infty} \dfrac{b}{e^b}$ L'Hôpital's Rule
  - limits algebraically, numerically and graphically
- $\lim_{k\to\infty} \vec{x}_k = \lim_{k\to\infty} \underbrace{c_1 \left(\frac{11}{10}\right)^k \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}}_{\text{dominant}} + c_2 \left(\frac{7}{10}\right)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$



http://notengofuerzasparararendirme.com/las-cosas-por-su-nombre/where_is_the_limit_petit/

Long-term Behavior of Foxes and Rabbits

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$$\lim_{k\to\infty} \vec{x}_k = \lim_{k\to\infty} \underbrace{c_1 \left(\tfrac{11}{10}\right)^k \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}}_{\text{dominant}} + c_2 \left(\tfrac{7}{10}\right)^k \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

In the longterm, for most starting positions, the system
(circle one): dies off, stabilizes, grows
as the line with equation
$y =$ ___ corresponding to dominant eigenvector ___ and
population ratios ___:___ with long-term rate of change
dominant eigenvalue-1 and as a percentage that is ___, except
if the coefficient of ___ equals 0, then the system
(circle one): dies off, stabilizes, grows
corresponding to ___.

Long-term Behavior of Foxes and Rabbits

$$\begin{bmatrix} \text{Fox}_k \\ \text{Rabbit}_k \end{bmatrix} = \begin{bmatrix} \frac{6}{10} & \frac{4}{10} \\ -\frac{125}{1000} & \frac{12}{10} \end{bmatrix} \begin{bmatrix} \text{Fox}_{k-1} \\ \text{Rabbit}_{k-1} \end{bmatrix}$$

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In the longterm, for most starting positions, the system grows as the line with equation $y = \frac{5}{4}x$ corresponding to dominant eigenvector $\begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}$ and population ratios 4:5 with long-term rate of change dominant eigenvalue-1 and as a percentage that is 10%, except if the coefficient of c_1 equals 0, then the system dies off corresponding to $\lambda = \frac{7}{10}$.

Owls and Wood Rats

In a redwood forest, wood rats may provide up to 80% of the diet of spotted owls. Let x=owls, y=wood rats

www.nps.gov/orca/learn/nature/images/mammal_1.jpg

For $A = \begin{bmatrix} \frac{21}{40} & \frac{3}{20} \\ -\frac{3}{16} & \frac{39}{40} \end{bmatrix}$, in Maple we execute

```
A := Matrix([[21/40,3/20],[-3/16,39/40]]);
Eigenvectors(A);
```

and obtain the output $\begin{bmatrix} \frac{3}{5} \\ \frac{9}{10} \end{bmatrix}, \begin{bmatrix} 2 & \frac{2}{5} \\ 1 & 1 \end{bmatrix}$ Notice that the eigenvectors span $\mathbb{R}^2$. Write the eigenvector decomposition.

Owls and Wood Rats

$\begin{bmatrix} \frac{3}{5} \\ \frac{9}{10} \end{bmatrix}$, $\begin{bmatrix} 2 & \frac{2}{5} \\ 1 & 1 \end{bmatrix}$

$$\vec{x}_k = c_1(\tfrac{3}{5})^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2(\tfrac{9}{10})^k \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$$

What does the trajectory look like for an initial vector in quadrant 1 that does not begin on either eigenvector?

a) dies off to the origin asymptotic to one eigenvector ($|\text{dominant eigenvalue}| < 1$)

b) grows asymptotic to one eigenvector ($|\text{dominant eigenvalue}| > 1$)

c) comes in parallel to one eigenvector with smaller and smaller contributions until we hit the other (dominant eigenvalue = 1)

Owls and Wood Rats

$\begin{bmatrix} \frac{3}{5} \\ \frac{9}{10} \end{bmatrix}, \begin{bmatrix} 2 & \frac{2}{5} \\ 1 & 1 \end{bmatrix}$

$$\vec{x}_k = c_1\left(\tfrac{3}{5}\right)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\left(\tfrac{9}{10}\right)^k \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$$

For most initial conditions, what happens to the system in the longterm?

a) populations die off in the ratios of 2 owls to 1 wood rat
b) populations die off in the ratios of 1 owl to 2 wood rats
c) populations die off in the ratios of 2 owls to 5 wood rats
d) populations die off in the ratios of 5 owls to 2 wood rats
e) other longterm behavior

Owls and Wood Rats

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For most initial conditions, the rate of die off in the longterm is 10% each year = 1 - dominant eigenvalue.

Sketch a trajectory diagram

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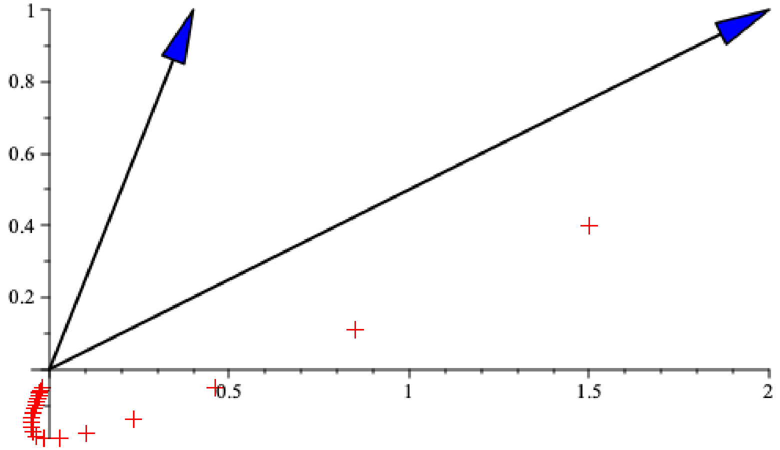

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Trajectory Diagrams and Longterm Behavior

- key is to know which eigenvalue stretches further
  `https://www.geogebra.org/m/nfvyhewj`
- trajectory diagram: create both eigenvectors using standard mathematical axes: $y = 1$ from keeping variables missing pivots free, so the vectors have the same height
- starting value in the 1st quadrant not on either eigenvector
- we approach dominant eigenvector in long run and eigenvalue tells us whether we have die off (magnitude less than 1), growth (magnitude greater than 1) or stability (equal to 1) in the long run
- never cross any line $\lambda\vec{x}$ an eigenvector is on: $A\vec{x} = \lambda\vec{x}$ means if we are on an eigenspace then we can't move off
- rate of change: difference of dominant eigenvalue and 1
- ratio of populations: dominant eigenvector

Sustainability

In the original matrix p is called a *predation parameter*: $\begin{bmatrix} \frac{21}{40} & \frac{3}{20} \\ -p & \frac{39}{40} \end{bmatrix}$. Find a value of p so that the populations tend towards constant levels (stability).

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plug in $\lambda = 1$:

0 = determinant $(A - \lambda I) =$ determinant $(A - 1 \cdot I)$

= det($\begin{bmatrix} \frac{21}{40} & \frac{3}{20} \\ -p & \frac{39}{40} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$) = $\begin{vmatrix} \frac{21}{40} - 1 & \frac{3}{20} \\ -p & \frac{39}{40} - 1 \end{vmatrix} = \frac{19}{1600} + \frac{3}{20}p$

$p = -\frac{19}{240}$

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$p = -\frac{19}{240}$

A := Matrix([[21/40,3/20],[19/240,39/40]]);

Eigenvectors(A); $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -6 & \frac{6}{19} \\ 1 & 1 \end{bmatrix}$

Sustainability

A := Matrix([[21/40,3/20],[19/240,39/40]]);

Eigenvectors(A); $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -6 & \frac{6}{19} \\ 1 & 1 \end{bmatrix}$

What are the relative population sizes in the longterm for most initial conditions? [owls (x-value) and wood rats (y-value)]

a) -6 owls to 1 wood rat
b) 1 owl to -6 wood rats
c) 6 owls to 19 wood rats
d) 19 owls to 6 wood rats
e) none of the above

Sustainability

A := Matrix([[21/40,3/20],[19/240,39/40]]);

Eigenvectors(A); $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -6 & \frac{6}{19} \\ 1 & 1 \end{bmatrix}$

The eigenvector decomposition:

$$\begin{bmatrix} \text{Owls}_k \\ \text{Wood rats}_k \end{bmatrix} = \left(\tfrac{1}{2}\right)^k \begin{bmatrix} -6a_1 \\ a_1 \end{bmatrix} + \begin{bmatrix} \frac{6}{19}a_2 \\ a_2 \end{bmatrix}$$ like geom of $t\vec{v}_1 + \vec{v}_2$

Sustainability

A := Matrix([[21/40,3/20],[19/240,39/40]]);

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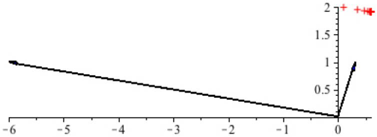
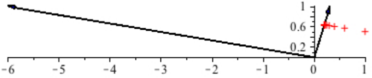
Trajectory diagrams:





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Another System

$$\vec{x}_k = a_1(\tfrac{2}{5})^k \begin{bmatrix} -2 \\ 1 \end{bmatrix} + a_2(\tfrac{4}{5})^k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Which of the following is true about the long-term trajectory of $\vec{x}_k$ for most starting positions?

a) the second population dies off along $y = \frac{1}{2}x$ but the first doesn't

b) both populations die off along $y = \frac{1}{2}x$ in the ratio of 2:1

c) other

Another System

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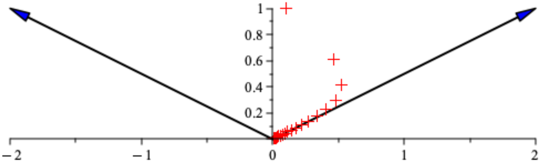

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rate of change in long run is $1 - \frac{4}{5} = \frac{1}{5} = 20\%$ die off

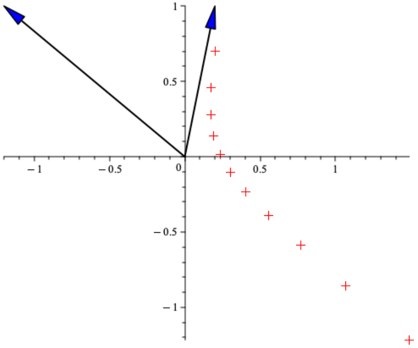
Two Predators

Execute the two-predator system in Maple and respond to the following questions:

- Explain why Maple's eigenvectors span all of $\mathbb{R}^2$ using either an argument using determinant and the what makes a matrix invertible theorem or the definition of span by augmenting with a generic vector and reducing
- If it exists, write the eigenvector decomposition for A?
- For most initial conditions, what ratio do the populations tend to in the long run and why?
- What is the yearly rate of change and why?
- Sketch the two eigenvectors and a trajectory with a starting position in quadrant 1 not on either eigenvector showing the long-term behavior.

Two Predators

$\vec{x}_k = a_1(\frac{7}{5})^k \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix} + a_2(\frac{7}{10})^k \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}$ growth asymptotic to $y = -\frac{5}{6}x$, $-6 : 5$ corresponding to dominant eigenvector, growth rate of change $\frac{7}{5} - 1 = 40\%$ from the dominant eigenvalue



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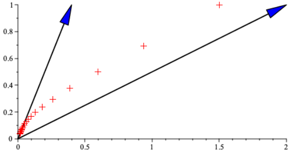
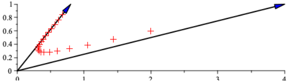
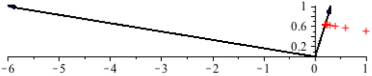
More Complicated Dynamics

All prior examples were linear in the limit:

$|\lambda| < 1$ die off 

$|\lambda| > 1$ growth 

$|\lambda = 1|$ stability 

parallel to weaker eigenvector until we stabilize to dominant one

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What do you call a young eigensheep?

https://cdn.vectorstock.com/i/composite/41,11/sheep-cartoon-vector-894111.jpg

What do you call a young eigensheep?
lamb, duh

λ!