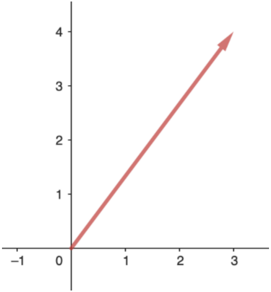
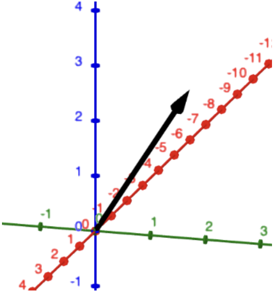
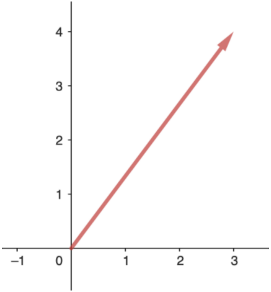
6.1: Inner Product, Length and Orthogonality

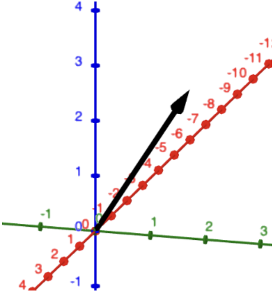




- length (norm) of $\vec{u} = ||\vec{u}||$
- $\left|\left|\begin{bmatrix}3\\4\end{bmatrix}\right|\right| = \sqrt{3^2 + 4^2}$

6.1: Inner Product, Length and Orthogonality





- length (norm) of $\vec{u} = ||\vec{u}||$
- $\left|\left|\begin{bmatrix}3\\4\end{bmatrix}\right|\right| = \sqrt{3^2+4^2}$ $\quad$ $\left|\left|\begin{bmatrix}1\\2\\3\end{bmatrix}\right|\right| = \sqrt{1^2+2^2+3^2}$
- Note in ASULearn sqrt(14)
- Why not defined? $\vec{v}_{n\times 1}\vec{v}_{n\times 1}$

- dot product of two vectors in $\mathbb{R}^n$

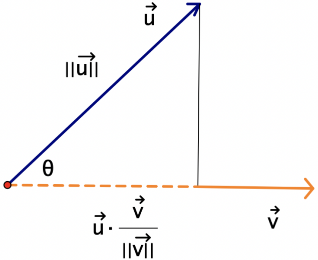
$$\vec{u} \cdot \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

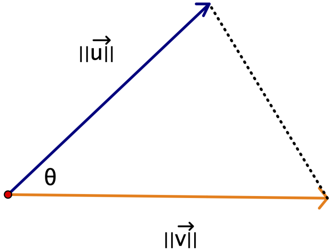
- as a matrix multiplication $\vec{u}^T \vec{v} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

- dot product of two vectors in $\mathbb{R}^n$

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

- as a matrix multiplication $\vec{u}^T \vec{v} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$
- length (norm) of $\vec{u} = ||\vec{u}|| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\vec{u}^T \vec{u}}$





- dot product $\vec{u} \cdot \vec{v}$ also is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them.

Orthogonality, Length, and Unit Vectors

- Apply $\vec{u} \cdot \vec{v}$ is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them, so if $\vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 0$ for nonzero vectors then angle is

Orthogonality, Length, and Unit Vectors

- Apply $\vec{u} \cdot \vec{v}$ is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them, so if $\vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 0$ for nonzero vectors then angle is $\frac{\pi}{2}$ or similar. Two vectors are orthogonal if right angle between them.
- Are $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ orthogonal?

Orthogonality, Length, and Unit Vectors

- Apply $\vec{u} \cdot \vec{v}$ is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them, so if $\vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 0$ for nonzero vectors then angle is $\frac{\pi}{2}$ or similar. Two vectors are orthogonal if right angle between them.
- Are $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ orthogonal? Yes as
  $\vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 1 \cdot -3 + 2 \cdot 0 + 3 \cdot 1 = 0$
- Compute the length of $\vec{v}$ and compare it to the length of $\vec{u}$. Which vector is longer?

Orthogonality, Length, and Unit Vectors

- Apply $\vec{u} \cdot \vec{v}$ is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them, so if $\vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 0$ for nonzero vectors then angle is $\frac{\pi}{2}$ or similar. Two vectors are orthogonal if right angle between them.
- Are $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ orthogonal? Yes as $\vec{u}^T\vec{v} = \vec{u} \cdot \vec{v} = 1 \cdot -3 + 2 \cdot 0 + 3 \cdot 1 = 0$
- Compute the length of $\vec{v}$ and compare it to the length of $\vec{u}$. Which vector is longer?
- $\frac{\vec{u}}{||\vec{u}||} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$ Next, create a unit vector in the same direction as $\vec{v}$ by dividing by the length.
- Maple

Rotation counterclockwise by θ

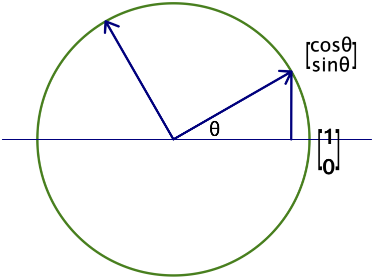
$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ with columns: $\vec{u} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \vec{v} = \begin{bmatrix} \text{-}\sin(\theta) \\ \cos(\theta) \end{bmatrix}$

What is the angle between the vectors?

Rotation counterclockwise by θ

$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ with columns: $\vec{u} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$

What is the angle between the vectors?



- negative reciprocal slope $\frac{\sin(\theta)}{\cos(\theta)}$ versus $\frac{\cos(\theta)}{-\sin(\theta)}$ or $\vec{u} \cdot \vec{v} = \cos(\theta)(-\sin(\theta)) + \sin(\theta)\cos(\theta) = 0$
- $||\vec{u}|| = \sqrt{\vec{u}^T \vec{v}} = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}}$

Geometric Transformations of $\mathbb{R}^2$

Rotation: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Dilation: $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

Reflection Matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

Reflections: $y = x$ line: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Geometric Transformations of $\mathbb{R}^2$

Rotation: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Dilation: $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

Reflection Matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

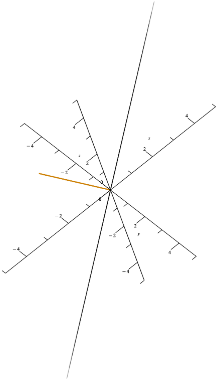
Reflections: $y = x$ line: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Projections: y=x line: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ y-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Horizontal Shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Vector of Coefficients of Consistency Equations for Span

$$\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 3 & b_1 \\ 1 & 3 & -2 & b_2 \\ 5 & 4 & 1 & b_3 \end{bmatrix}$$

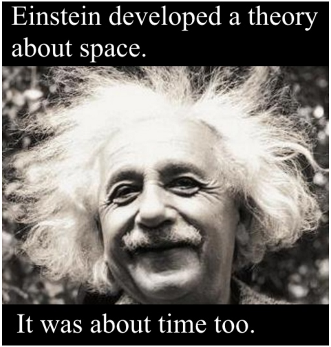


Inner Products and Norms

- Here inner product $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$ induces norm or metric on the space $||\vec{u} - \vec{v}||$ is distance between vectors as in 1.3

Inner Products and Norms

- Here inner product $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$ induces norm or metric on the space $||\vec{u} - \vec{v}||$ is distance between vectors as in 1.3
- Generalized inner products for nonlinear/non-Euclidean spaces satisfy axiomatic properties like distributivity, pulling out scalars, positive definite condition



http://jokerific.com/wp-content/uploads/2014/08/einstein-space-time-theory-joke.jpg

Applications

- 2.7 computer graphics, 6.5 least squares, 6.6 machine learning
- relativity and differential geometry

  $\cos\theta = \frac{w^T g_{ij} v}{|v||w|}$, spacetime interval: $|v| = \sqrt{v^T g_{ij} v}$

  $$\text{Minkowski: } g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

  $$\text{wormhole: } g_{ij} = \begin{bmatrix} -1 + \frac{r_0}{r} - \frac{\epsilon}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r_0}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

  where r is radial, θ are angular, ϵ is electric charge, r_0 is smallest radius of throat
- work, arc length