Dark Matter Accretion and the Hessian Matrix

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IN-CLASS CONNECTIONS

The determinant of a matrix is the main topic from MAT 2240 that will aide in the understanding of research on dark matter accretion. Per the course lectures, not every matrix has a determinant.[1] In order for a determinant to be calculated from a given matrix, the matrix must be square (dimensions *n x n*).[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Figure 1. A generic matrix with dimensions *2 x 2* fits the criteria for calculating a determinant.

A standard determinant calculation for a generic matrix, such as Figure 1, follows the form of *(ad-bc)*.[1] This is also the product of the terms in the off-diagonal subtracted from the product of the terms on the main diagonal. For matrices of higher dimensions, other methods are needed for determinant calculation. The method of Laplace expansion works for determinant calculation for matrices of any dimension.[1]

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Figure 2. A Laplace expansion for a generic *3 x 3* matrix. This method holds for matrices of higher dimensions, but for the purposes of this paper, a *3 x 3* example will suffice.[2]

The Laplace expansion is performed by expanding along a certain row or column of a matrix. The first term in the row or column (term i,j) is multiplied by the determinant of the matrix with the ith row and jth column deleted, as seen in Figure 2. This product is then summed with the next term in the original row/column previously decided on. Figure 2 shows Laplace expansion along the first column of a *3 x 3* matrix. For a triangular matrix, (a matrix with all zeroes above or below the main diagonal) the determinant is simply the products of the terms on the main diagonal.[1]

Values of determinants can describe certain things about matrices, like invertibility, for example. A matrix with a nonzero determinant is invertible.[1] Because a matrix must be square in order for a determinant to be taken, there are implications about the vectors within the square matrix. From the class-popular Theorem 8, the matrix invertibility theorem, we also know that a matrix with a nonzero determinant has linearly independent vectors and that those vectors span the entire space which they occupy.[1] The determinant of a *2 x 2* matrix yields the area of the parallelogram spanned by the two vectors in the matrix. The determinant of a *3 x 3* matrix yields the volume of the parallelepiped cast by the three vectors.[1] One can see how determinants provide useful information when conducting experiments that involve mass or density distributions, such as the one discussed in the proceeding sections of this paper. In-class examples of determinants, their meanings, and calculations can be found in Exercise 3.3, as well as in Problem Set 4.

Determinants provide insight into another intrinsic property of matrices: eigenvectors and eigenvalues. Eigenvalues of a matrix are values that satisfy the following equation:

$$det\,(A - \lambda I) = 0 \qquad (1)$$

In equation (1), det is the determinant function, A is the matrix whose eigenvalues are being calculated, λ represents the eigenvalues(s) to be calculated, and I represents the identity matrix. This is called the characteristic equation, and is used to determine eigenvalues for square matrices.[1] Eigenvalues are unique in that they make scalar multiplication equivalent to matrix multiplication. This unique property is shown by equation (2).

$$A\vec{x} = \lambda\vec{x} \qquad (2)$$

Equation (2) highlights the uniqueness of eigenvalues. Eigenvalues have corresponding eigenvectors that hold special properties of their own. Eigenvectors are vectors in space that remain unchanged when the matrix being studied is scaled by one of its eigenvalues.[3] Per the course material, eigenvectors and eigenvalues can be used to describe population trends. The familiar "fox and rabbit" problem hinges entirely on eigenvectors, their corresponding eigenvalues, and how the population trends according to both of those combined. These population trends can be described in an eigenvector decomposition equation, as in equation 3.

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = a_1\lambda_1^k \begin{pmatrix} e_{x1} \\ e_{y1} \end{pmatrix} + a_2\lambda_2^k \begin{pmatrix} e_{x2} \\ e_{y2} \end{pmatrix} \quad (3)$$

In equation 3, a_1 and a_2 are constants that are determined from the initial conditions of the population or trending system. The λ variables are the eigenvalues of the population matrix, and the **e** vectors are the corresponding eigenvectors to their specific eigenvalues. The way that this mathematical system predicts trend is by taking limits at infinity. When the k term is allowed to approach infinity, long-term behavior for the system in question becomes discernible. If the eigenvalues of the system are not equal, one value will be inherently dominant over the other.[1] This dominant eigenvalue forces the system to trend toward its corresponding eigenvector asymptotically in most cases, and rates of increase or decrease can be determined from then on.[1] Systems will grow, stabilize, or die off in accordance to their dominant eigenvalue. Eigenvalues with absolute values less than one will die off. Thos equal to one will stabilize, and those greater than one will see overall growth. Eigenvalues are not always real values. Some matrices, like most rotation matrices, have complex eigenvalues.

Eigenvalues have another important use when considering objects in three dimensions. When considering surfaces in three-space, there exists a special matrix called the Hessian matrix, which is a matrix whose entries is all the partial derivatives for a certain multivariate function.[4] By calculating the eigenvalues of a Hessian matrix, one can discern certain points on the surface in question to be maxima, minima, or saddle points.[4] Such information can prove to be extremely useful when considering surfaces in three dimensions, such as galaxies or masses on a cosmological scale. The importance of the Hessian matrix hinges entirely on the process of calculating determinants, eigenvalues, and, in the case of this experiment summary, understanding how eigenvalues and eigenvectors influence the trajectory of a trending system. Without these core topics covered in MAT 2240, this sort of large-scale cosmological study could never have yielded any useful or practical information.

APPENDIX A

Relevant Examples from the Course

```
> A3:=Matrix([[1,1,1,1,1],[1,2,2,2,2],[1,2,3,3,3],
  [1,2,3,4,4],[1,2,3,4,5]]); Determinant(A3);
```

$$A3 := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Figure 3. A mathematics program like Maple allows almost instantaneous computation of determinants of matrices with larger dimensions than the familiar *2x2* or *3x3*. This example, although relatively simplistic, is conceptually crucial to understanding the dark matter accretion study.[5] This sample determinant calculation was taken from the first question in Problem Set 4.

```
> A:=Matrix([[0,16/10],[3/10,8/10]]);
```

$$A := \begin{bmatrix} 0 & \frac{8}{5} \\ \frac{3}{10} & \frac{4}{5} \end{bmatrix}$$

```
> Eigenvectors(A); evalf(Eigenvectors(A));
```

$$\begin{bmatrix} -\frac{2}{5} \\ \frac{6}{5} \end{bmatrix}, \begin{bmatrix} -4 & \frac{4}{3} \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -0.4000000000 \\ 1.200000000 \end{bmatrix}, \begin{bmatrix} -4. & 1.333333333 \\ 1. & 1. \end{bmatrix}$$

Figure 4. This sample eigenvector calculation is the beginning of an eigenvector decomposition equation. Although the equations used in the dark matter study are for more complex, the same ideas apply.[5] Discussion in class focused mainly on the trajectory of populations in the long term, given a few starting parameters. In the accretion study, trajectory diagrams are vital to understanding the final results. For example, as we have seen before in these sorts of problems, eigenvalues have their unique, corresponding eigenvectors. The same holds true in the dark matter study. The eigenvectors calculated, however, represents lines of slow or rapid collapse rate of matter instead of trending population.[5] This example problem was taken from the third question in Problem Set 4.

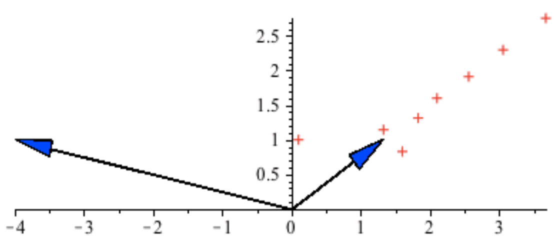

Figure 5. This is an extension of the same sample problem as that in Figure 4. This image highlights another crucial concept of eigenvectors. The cross points represent how the hypothetical population trends with time. There is an observable, asymptotic trend to the eigenvector in quadrant 1. This image shows how a system's behavior can trend toward what is called its "dominant" eigenvector, in most cases. This concept of trajectory proves to be the focal point of understanding the results from the dark matter accretion study.[5]

Most of the in-class connections for this project stemmed from material covered in the last few weeks. Although computable by-hand, many of the sample problems and exercises can be done in a computational software, such as Maple or Mathematica. Using such a software is favorable for the sake of time, but it is important to understand the mechanics, algorithms, and concepts that go into these computations.

EXTENSION OF CLASS MATERIAL

In a study conducted by Xi Kang and Peng Wang of the University of the Chinese Academy of Science, linear algebra was used in a familiar (to us) fashion to understand natural phenomena on a macroscopic level. This research was done via computer simulation using the WMAP7 data* as parameters for the virtual cosmos. Two simulations were run simultaneously. One simulation was a low-resolution, low mass environment. The other was the opposite: a high resolution, high mass environment. The advantage of running dual simulations as described above, is that multiple concurrent runs allows data to be taken for both high-mass and lower-mass systems, two common things that exist in the cosmos.[5] Studies of our own galaxy have shown that the Milky Way's rotation speed is too fast given its measurable amount of mass.[6] This implies that there is matter present that is not being detected, hence the term "dark matter". The simulation used in this study accounts for this dark matter by using previously-theorized algorithms.

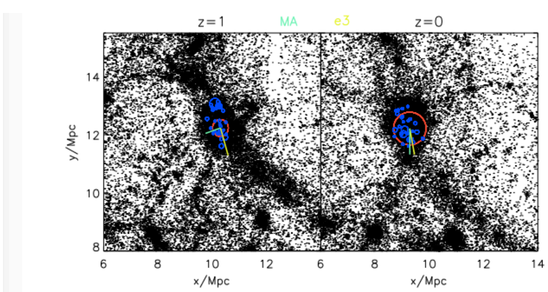

Figure 6. This screenshot from the study's simulation efforts highlights the calculations done by the simulation software. The large (red) circles in the middle represent the virial radius of the collapsing system, while the smaller circles (blue) represent the dark matter halos forming about the collapsing mass. Some small circles appear to be inside the red circles due to this image being a two dimensional representation of a three dimensional study.[5]

This experimental study of dark matter hinges entirely on the Zel'dovich theory. This theory states that when matter is collapsing, the halo will tend to when of several cosmological classes of mass distribution, depending on the eigenvalues of its Hessian matrix.[5] The Hessian matrix, in this instance is the matrix filled with all the partial derivatives of a function that represents the mass or density distribution being studied. The number of positive eigenvalues indicates what type of mass the halo will condense to.[5] For example, one positive eigenvalue leads to a filament, two positive eigenvalues leads to a sheet, and no positive eigenvalues leads to a cluster.[5]

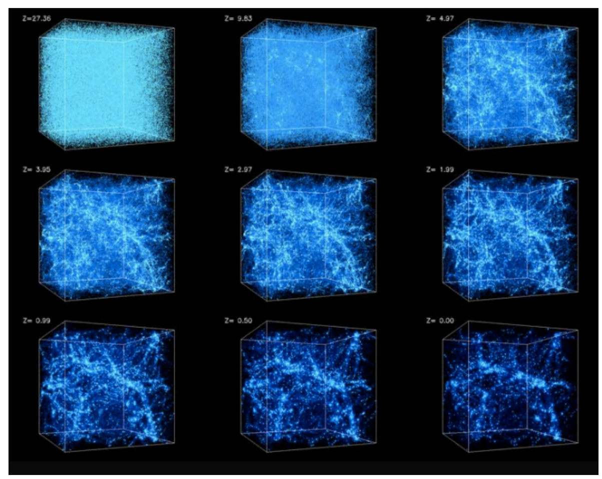

Figure 7. This image shows a trend of collapsing cosmological mass. This picture shows examples of sheets, filaments, clusters, and voids.[7] Available at: http://www.astronomynotes.com/galaxy/filaments-uchicago.jpg

The final results of this extended computer simulation established some relationship between collapsing masses, the dark matter surrounding them, and the eigenvectors of the whole system. For example, after accretion has nearly completed, it was seen that the major axis of the dark matter halo was well aligned with the least-compressed direction of the entire system (the eigenvalue with the smallest magnitude).[5] Likewise, it was also seen that subhalos were more aligned with the major axis of the host halo when the system was more massive. This has interesting implications into the idea of dominant eigenvectors. In the more massive systems, the alignment of matter halos was more linear, i.e., the eigenvalue with the largest magnitude was dominant. However, in the less massive systems, the alignment of matter halos was closer to perpendicular, compressing along the eigenvector with the weakest corresponding eigenvalue.[5] The study calls this the "fast collapse vs. slow collapse", which makes sense conceptually. If a system has more mass, it will collapse faster and more uniformly then a system with less mass.

APPENDIX B

*WMAP7 was an experiment that successfully mapped out an estimated mass distribution on a universal scale. This was done by a space probe that emitted radio waves to detect any sort of heat coming from matter scattered throughout the universe.

REFERENCES

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