> `with(LinearAlgebra):`

Geometry of Eigenvectors

> `A:=Matrix([[0,1],[1,0]]);`

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We would like to know when A $\begin{bmatrix} x \\ y \end{bmatrix}$

ends up on the same line through the origin as the original vector started on. We apply A and test out different points,

like (1,0), (1,1), (0,1),(-1,1) and draw 4 different pictures - draw an input vector, say in red or dashed, and an output vector, say in blue or solid. We are then looking for vectors whose input and output stay on the same line through the origin, and these are called eigenvectors.
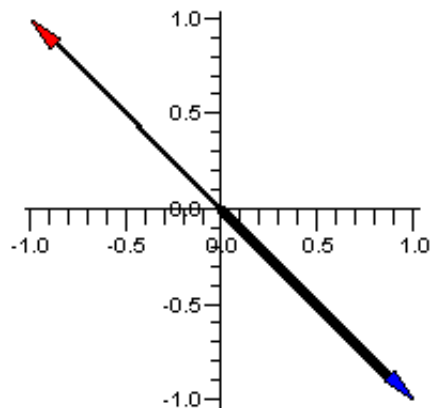

> `u := Vector([1,1]);`

> `v := Vector([-1,1]);`

$$u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v := \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Since A takes (x,y) to (y,x), then it is a reflection in the line y=x. Any vector on the line y=x (with a basis element of (1,1)) gets fixed, and any vector on the line y=-x (with a basis element of (1,-1) stays on that line since it gets flipped. Sometimes it is easier to visualize eigenvectors by putting the tail of the output vector Ax at the head of the input vector x (again, the input vectors have thin shafts and the output vectors thick shafts):
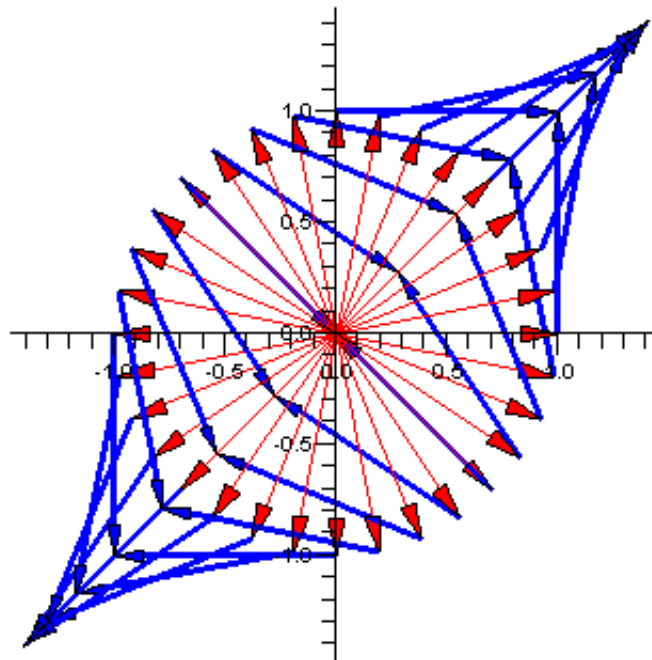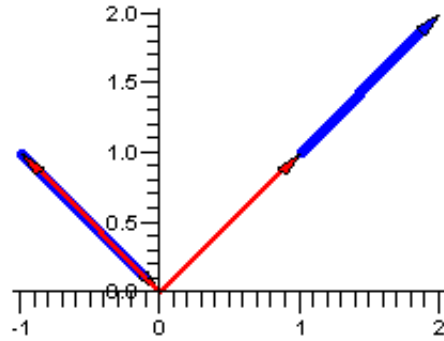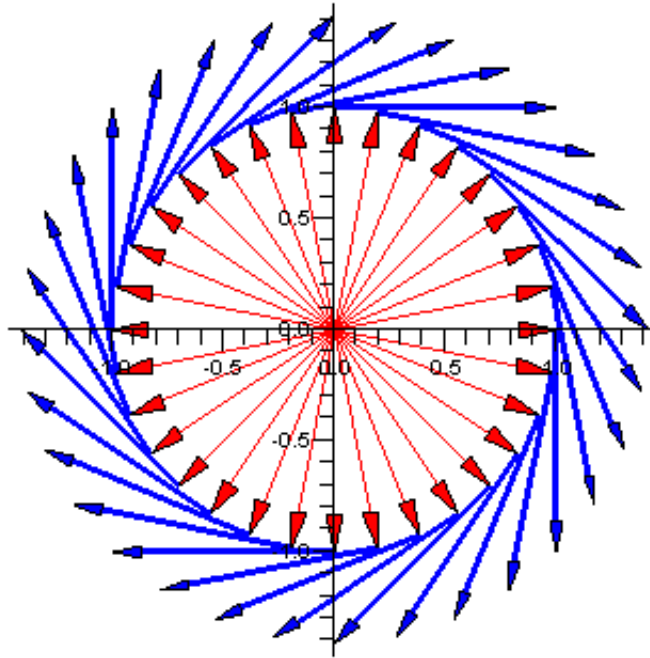

Let's look at another example:

```
> M:=Matrix([[0,1],[-1,0]]);
```

$$M := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This matrix takes (x,y) to (y,-x). Test out different points, like (1,0), (1,1), (0,1),(-1,1) and draw an input vector, say in red or dashed, and an output vector, say in blue or solid. We are then looking for vectors whose input and output stay on the same line through the origin, and these are the eigenvectors.


Notice that this is a rotation matrix like we discussed previously - it fits into the form Matrix([[cos(theta),-sin(theta)],[sin(theta),cos(theta)]]), by taking theta= - 90 degrees, and so every vector is rotated by 90 degrees off of the line it started on, and so there are no eigenvectors.

Let's look at another example:

```
> M:=Matrix([[-1,0],[0,-1]]);
```

$$M := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This matrix takes (x,y) to (-x,-y). Test out different points, like (1,0), (1,1), (0,1),(-1,1) and draw an input vector, say in red or dashed, and an output vector, say in blue or solid. We are then looking for vectors whose input and output stay on the same line through the origin, and these are the eigenvectors.
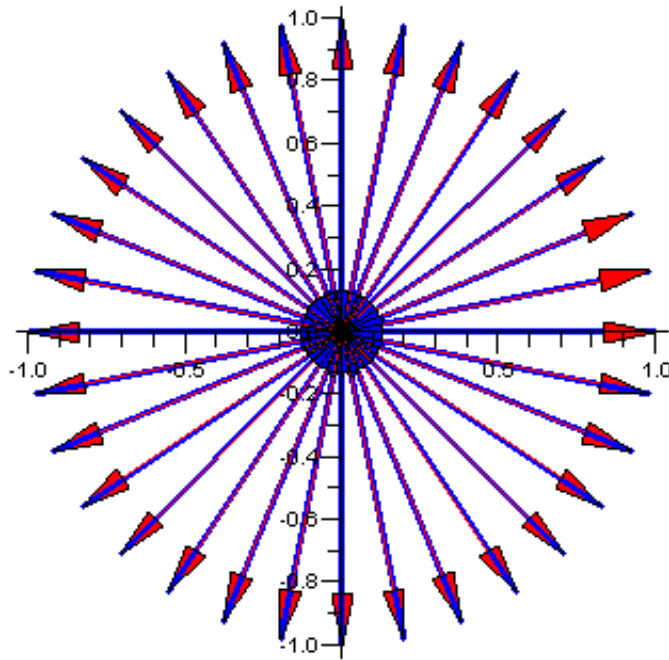
This is rotation by 180 degrees and every vector is an eigenvector since it stays on the same line through the origin by being flipped.

Finally let's examine:

```
> M:=Matrix([[1/2,1/2],[1/2,1/2]]);
```

$$M := \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix}$$

Test out different points, like (1,0), (1,1), (0,1),(-1,1) and draw an input vector, say in red or dashed, and an output vector, say in blue or solid. We are then looking for vectors whose input and output stay on the same line through the origin, and these are the eigenvectors.
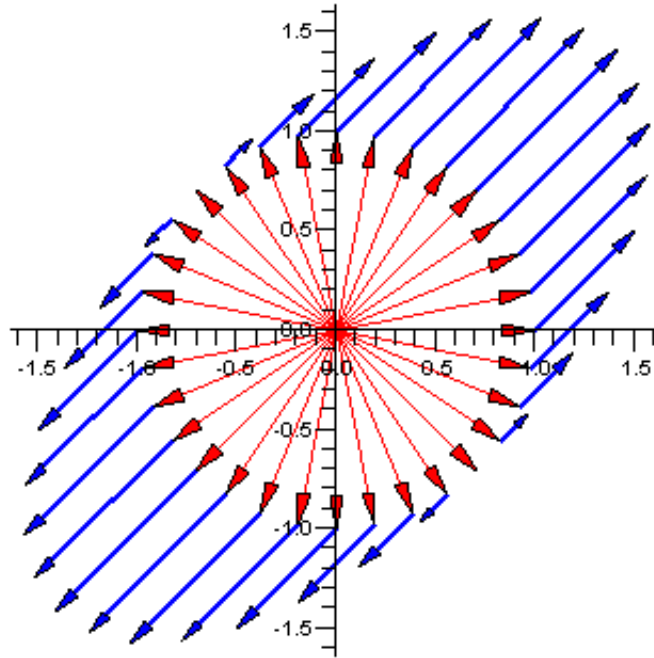
This is a projection matrix, that takes (x,y) to (1/2 x + 1/2 y, 1/2 x +1/2 y), ie projected onto the y=x line. If we look at light rays perpendicular to the y=x line, this matrix gives us the shadow a vector makes onto the y=x line, which has many applications in mathematics and physics. Any vectors on the lines y=x and y=-x, with basis representatives (1,1) and (-1,1), are eigenvectors here [Notice that (-1,1) or anything else on that line gets sent to (0,0) which is still on the same line through the origin, and anything on y=x, like (1,1) gets fixed by M.]