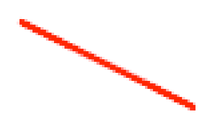
Select Practice Problems 4.4-4.6

```
> with(LinearAlgebra): with(plots):
```

Maple Problem 1

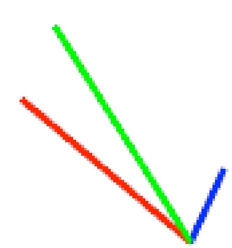
Let v0 = (2,-1,1) and let L=Span of (2,-1,1) = {t (2,-1,1) where t is real}. Notice L is a line through the origin in R3 and we can graph it in Maple as follows:

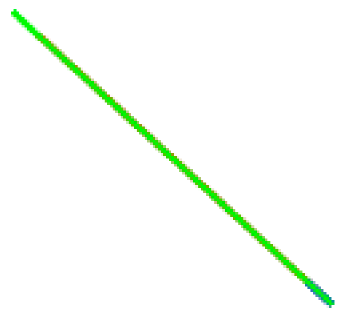
```
> a := spacecurve({[2*t, -1*t, 1*t, t = 0..1]}, color = red) :
> display(a);
```



Part 1: Find a vector w0 so that {v0, w0} is a basis for some plane P0 through the origin containing L.
We wish to choose something that is off the original line, ie anything that is not a multiple of (2,-1,1). So let's choose (1,0,0)
Part 2: Find a vector w1 not equal to w0 so that {v0, w1} is also a basis for the same plane P0
We could choose a multiple of (1,0,0), like (2,0,0), or any linear combination of (2,-1,1) and (1,0,0) that does not lie on the (2,-1,1) line. Let's choose the sum of the two vectors, ie (3,-1,1), which is the diagonal of the parallelogram formed by them.
Part 3: In Maple, use commands similar to the spacecurve and display commands above to show that all three vectors lie in the same plane [use different colors like black, blue, green..., and one display command like display(a,b,c);]

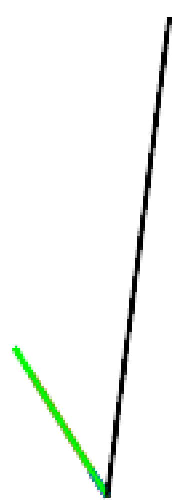
```
> b := spacecurve({[1*t, 0*t, 0*t, t = 0..1]}, color = blue) :
> c := spacecurve({[3*t, -1*t, 1*t, t = 0..1]}, color = green) :
> display(a, b, c);
```





Notice that I have turned the plane to view it head-on to see that they all lie there.
Part 4: Describe all the vectors w for which {v0, w} is a basis for the same plane P0 [Hint - think about what linear combinations you can use].
We can choose any linear combination of (2,-1,1) and (1,0,0) that does not lie on the (2,-1,1) line, ie t (2,-1,1) + s (1,0,0), where s and t are any real numbers except that s cannot equal 0.
Part 5: Find a vector u0 so that {v0, u0} is a basis for a different plane P1 through the origin.
Here we must choose something outside the plane, ie something that is not a linear combination of (2, -1,1) and (1,0,0). I'll take (10,5,1).
Part 6: Add u0 to your graph from Part 3 to show it lies outside the plane.
I will graph the plane, and turn it head on to show that (10,5,1) sticks out.

```
> d := spacecurve({[10*t, 5*t, 1*t, t = 0..1]}, color = black) :
> display(a, b, c, d);
```



Maple Problem 2

Let W be the subspace of R^4 spanned by the vectors u1:=(1,2,3,4), u2:=(4,2,1,5), and u3:=(3,5,1,7). Determine if the vectors v:=(8,9,5,16) and w:=(7,2,1,3) are in W by setting up the augmented matrices and solving.
We put the vectors as columns and augment with v and then w.

```
> A := Matrix([[1, 4, 3, 8], [2, 2, 5, 9], [3, 1, 1, 5], [4, 5, 7, 16]]);
```

$$A := \begin{bmatrix} 1 & 4 & 3 & 8 \\ 2 & 2 & 5 & 9 \\ 3 & 1 & 1 & 5 \\ 4 & 5 & 7 & 16 \end{bmatrix} \quad (2.1)$$

> *ReducedRowEchelonForm*(*A*);

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.2)$$

Notice that v can be written as a linear combination of the 3 vectors - just add them together (ie c1=1, c2=1, and c3=1). We can tell that u1,u2,u3,v is not linearly independent from here already.

> *B* := *Matrix*([[1, 4, 3, 7], [2, 2, 5, 2], [3, 1, 1, 1], [4, 5, 7, 3]]);

$$B := \begin{bmatrix} 1 & 4 & 3 & 7 \\ 2 & 2 & 5 & 2 \\ 3 & 1 & 1 & 1 \\ 4 & 5 & 7 & 3 \end{bmatrix} \quad (2.3)$$

> *ReducedRowEchelonForm*(*B*);

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.4)$$

Here we get an inconsistent system so w is not a linear combination of the 3 vectors. We cannot tell whether u1,u2,u3,w is linearly independent, since u1 could still be a combination of u2, u3, and w.

Next, determine whether {u1, u2, u3, v} and {u1, u2, u3, w} are linearly independent sets by setting up the homogeneous equations and solving.

> *AH* := *Matrix*([[1, 4, 3, 8, 0], [2, 2, 5, 9, 0], [3, 1, 1, 5, 0], [4, 5, 7, 16, 0]]);

$$AH := \begin{bmatrix} 1 & 4 & 3 & 8 & 0 \\ 2 & 2 & 5 & 9 & 0 \\ 3 & 1 & 1 & 5 & 0 \\ 4 & 5 & 7 & 16 & 0 \end{bmatrix} \quad (2.5)$$

> *ReducedRowEchelonForm*(*AH*);

(2.6)

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.6)$$

> $BH := Matrix([[1, 4, 3, 7, 0], [2, 2, 5, 2, 0], [3, 1, 1, 1, 0], [4, 5, 7, 3, 0]]);$

$$BH := \begin{bmatrix} 1 & 4 & 3 & 7 & 0 \\ 2 & 2 & 5 & 2 & 0 \\ 3 & 1 & 1 & 1 & 0 \\ 4 & 5 & 7 & 3 & 0 \end{bmatrix} \quad (2.7)$$

> $ReducedRowEchelonForm(BH);$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (2.8)$$

We see there are infinitely many solutions with v, so the system is not linearly independent, but there is only the trivial solution with w so, in fact, it is linearly independent.

4.4 number 11

Determine whether S={(4,7,3),(-1,2,6),(2,-3,5)} spans R^3. We want to know whether any vector (u1,u2,u3) in R^3 can be written can be written as a linear combination of the vectors in S. So, we can set up the system
M *column vector(c1,c2,c3)=column vector(u1,u2,u3) to see if we always get a solution, where M is

```
> M:=Matrix([[4, -1, 2],[7,2,-3],[3,6,5]]);
```

$$M := \begin{bmatrix} 4 & -1 & 2 \\ 7 & 2 & -3 \\ 3 & 6 & 5 \end{bmatrix}$$

```
> Determinant(M);
```

$$228$$

Since the determinant is 228 which is not 0, then we know that the system has a unique solution column vector(c1,c2,c3)=(inverse of M)column vector(u1,u2,u3)
Thus every column vector (u1,u2,u3) can be written as a linear combination of the vectors in S, and so the set S spans R^3.

Augmented Matrix Method: Note that setting up the augmented matrix for the system is problematic because the =constants are unknown. We can use Gaussian though, as usual for k, or a,b, c problems:

```
> M:=Matrix([[4, -1, 2,u1],[7,2,-3,u2],[3,6,5,u3]]);
```

$$M := \begin{bmatrix} 4 & -1 & 2 & u1 \\ 7 & 2 & -3 & u2 \\ 3 & 6 & 5 & u3 \end{bmatrix}$$

```
> GaussianElimination(M);
```

$$\begin{bmatrix} 4 & -1 & 2 & u1 \\ 0 & \frac{15}{4} & -\frac{13}{2} & u2 - \frac{7}{4}\,u1 \\ 0 & 0 & \frac{76}{5} & u3 + \frac{12}{5}\,u1 - \frac{9}{5}\,u2 \end{bmatrix}$$

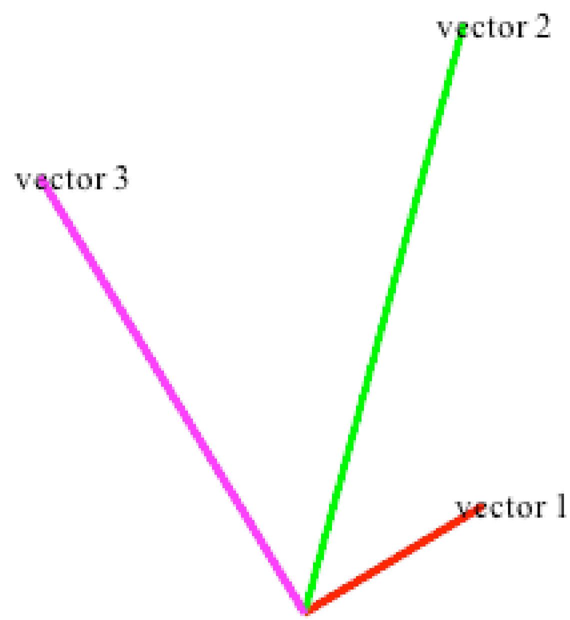
We can explain that z = (u3+12/5*u1-9/5*u2)/(76/5), and that backsub will work to give a unique solution (c1,c2,c3 coefficients) for each (u1,u2,u3).

Visualization We cannot graph the complete rows of the augmented matrix as equations of planes, as in Chapter 1, because u1, u2, and u3 are unknown. But we can visualize the column vectors of the coefficient matrix M as vectors in R^3, as in Chapter 4.

```
> a1:=spacecurve({[4*t,7*t,3*t,t=0..1]},color=red, thickness=2):

  a2:=textplot3d([4,7,3, `        vector 1`],color=black):
  b1:=spacecurve({[-1*t,2*t,6*t,t=0..1]},color=green, thickness=2):

  b2:=textplot3d([-1,2,6, `        vector 2`],color=black):
  c1:=spacecurve({[2*t,-3*t,5*t,t=0..1]},color=magenta, thickness=
  2):
  c2:=textplot3d([2,-3,5, `        vector 3`],color=black):
  display(a1,a2, b1,b2,c1,c2);
```



It takes some dynamic turning in Maple in order to convince yourself that the vectors do not all lie in the same plane. Hence they must span all of R^3.

4.4 number 26

To determine whether the set S={(1,0,0),(0,4,0),(0,0,-6),(1,5,-3)} is linearly independent or dependent, we want to form the equation M*column vector(c1,c2,c3,c4)=(0,0,0), and see if we have any non-trivial solutions. We can't use the inverse method on this since M is a 3x4 matrix, so we will form the augmented matrix

```
> M:=Matrix([[1,0,0,1,0],[0,4,0,5,0],[0,0,-6,-3,0]]);
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 5 & 0 \\ 0 & 0 & -6 & -3 & 0 \end{bmatrix}$$

```
> ReducedRowEchelonForm(M);
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{5}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Since we have 4 unknowns (c1,c2,c3,c4) and 3 consistent equations, we have an infinite number of solutions.
Hence the set is linearly dependent. For example, we could take c4=-1, c3=1/2, c2=5/4 and c1=1.
Then (1,5,-3) = (1,0,0) +5/4 (0,4,0)+1/2(0,0,-6).

4.4 number 53

The set {(1,2,3),(1,0,-2),(-1,0,2)} is linearly dependent but (1,2,3) cannot be written as a linear combination of (1,0,-2) and (-1,0,2). Why does this statement not contradict Theorem 4.8?

Theorem 4.8 requires that only one of the vectors be a linear combination of the others. In this case, (-1,0,2) = 0*(1,2,3) -1* (1,0,-2) and so one of the vectors is a linear combination of the others.

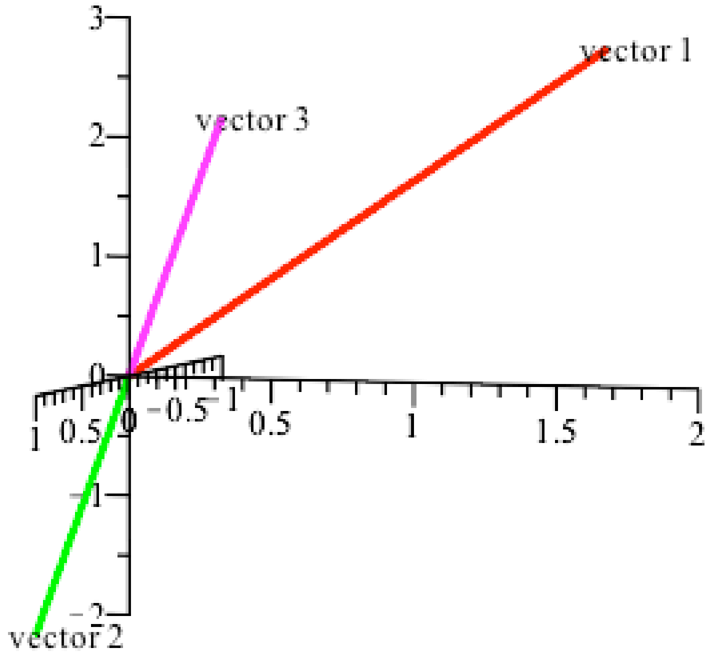
Below I've graphed the vectors in 3-space. It is clear that (1,0,-2) and (-1,0,2) lie on the same line through the origin because they are multiples of each other. Hence using them both is not an efficient way of representing a space. The set of three vectors spans a plane in R^3, but it is more efficient to just take 2 of them that are linearly independent to span that plane: {(1,2,3), (1,0,-2)}.

```
> a1:=spacecurve({[t,2*t,3*t,t=0..1]},color=red, thickness=2):

  a2:=textplot3d([1,2,3, `        vector 1`],color=black):
  b1:=spacecurve({[1*t,0,-2*t,t=0..1]},color=green, thickness=2):

  b2:=textplot3d([1,0,-2, `        vector 2`],color=black):
  c1:=spacecurve({[-1*t,0,2*t,t=0..1]},color=magenta, thickness=2):

  c2:=textplot3d([-1,0,2, `        vector 3`],color=black):
  display(a1,a2, b1,b2,c1,c2);
```



4.5 number 22

We know that R^2 has dimension 2, and so any set of 2 linearly independent vectors in R^2 is a basis for R^2. Notice that the two vectors are linearly independent since one is not a scalar multiple of the other. Hence this is a basis for R^2.

4.6 number 22

First we need to solve the system Ax=0. We will look at the augmented matrix for the system and use Gauss-Jordan:

```
> M:=Matrix([[4,-1,2,0],[2,3,-1,0],[3,1,1,0]]);
```

$$M := \begin{bmatrix} 4 & -1 & 2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

```
> ReducedRowEchelonForm(M);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Notice that this system has only one solution x=y=z=0. Thus, the solution space is {(0,0,0)} whose dimension is 0.

4.6 number 31

First we need to solve the system Ax=b. We will look at the augmeed matrix for this system and use Gauss-Jordan:

```
> M:=Matrix([[1,2,1,1,5,0],[-5,-10,3,3,55,-8],[1,2,2,-3,-5,14],[-1,
  -2,1,1,15,-2]]);
```

$$M := \begin{bmatrix} 1 & 2 & 1 & 1 & 5 & 0 \\ -5 & -10 & 3 & 3 & 55 & -8 \\ 1 & 2 & 2 & -3 & -5 & 14 \\ -1 & -2 & 1 & 1 & 15 & -2 \end{bmatrix}$$

```
> ReducedRowEchelonForm(M);
```

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & 0 & 6 & 2 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us infinitely many solutions x5=t, x4=-3+4t, x3=2-6t, x2=s, x1=1+5t-2s.

Part B: Now we rewrite the infinitely many solutions x= (1,0,2,-3,0) + s(-2, 1, 0,0,0) + t(5, 0,-6,4,1). The homogeneous solution is s(-2, 1, 0,0,0) + t(5, 0,-6,4,1) and these two vectors form a basis for the two dimensional homogeneous solution space. The particular solution is (1,0,2,-3,0).

4.6 number 29

Part A: The system Ax=b is inconsistent since the augmented matrix

```
> M:=Matrix([[3,-2,16,-2,-7],[-1,5,-14,18,29],[3,-1,14,2,1]]);
```

$$M := \begin{bmatrix} 3 & -2 & 16 & -2 & -7 \\ -1 & 5 & -14 & 18 & 29 \\ 3 & -1 & 14 & 2 & 1 \end{bmatrix}$$

reduces to

```
> ReducedRowEchelonForm(M);
```

$$\begin{bmatrix} 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which gives us no solutions since the last row gives us $0w+0x+0y+0z=1$, which is not possible.