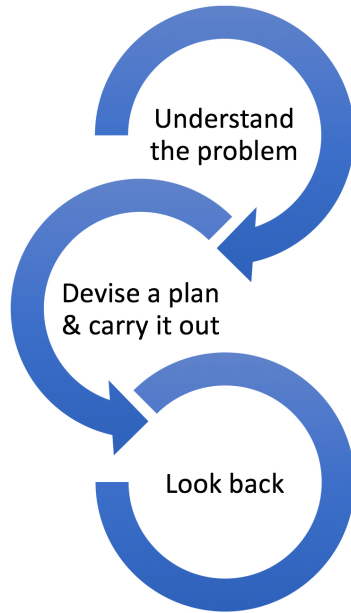


## Problem Set 2

**Instructions:** You may work alone or in a group of up to 2 people and turn in one per group but each group must complete and write up their work in their own words. Your group collates your problem set into one single PDF for submission in the ASULearn assignment. Feel free to talk to me (I am always happy to help!) or each other if you are stuck on this assignment, but be sure to acknowledge any sources or people, aside from your partner or me. If you know how to do a problem and are asked for help, try to give hints rather than the solution. Explain in your own words, execute your own adaptations to the Maple file **prs2s22.mw** (and include any code and output), create your own pictures (if any), and give any references you used (if any), including each other.



*Mathematics, you see, is not a spectator sport.* [George Polya, *How to Solve it*]

1. List your preferred name(s). If you are turning this in with a partner, list both names.

### 2. Invertible Matrix Theorem

Review the invertible matrix theorem in 2.3. Which of the equivalent statements do you find most useful or important? Explain your thoughts.

### 3. Column Space, NullSpace and Rank-Nullity Theorem

Let  $A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix}$ . To make connections between Maple and by-hand work for the column space and nullspace...

- What is our book's definition (or the glossary) of the column space of a matrix  $A$ , or  $\text{Col } A$ ?
- Use **prs2s22.mw** to reduce  $A$  to reduced row echelon form. Show or write the output.
- Write a basis for  $\text{Col } A$  from the pivot columns of  $A$ .
- What is the geometry of  $\text{Col } A$ ? Explain.

- (e) To look for another representation of Col  $A$  via all the  $\vec{b}$  vectors so that  $A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$  is consistent, use **prs2s22.mw** to reduce  $[A\vec{b}]$ , to row echelon form using Gaussian. Show or write the output.
- (f) Are there any inconsistent parts (like  $[0 \ 0 \ 0 \ 0 \ 0]$  combination of bs) to set equal to 0 and provide one or more algebraic representations that  $\vec{b}$  in the span satisfy—condition(s) on  $b_1, b_2, b_3$  and/or  $b_4$ ? If so, list any.
- (g) In **prs2s22.mw**, execute the ColumnSpace command to see the basis that Maple provides. Show or write the output.
- (h) Compare and contrast Maple's basis for the column space to the basis from the pivot columns in part (c)? Discuss.
- (i) Show that each vector in Maple's basis satisfies any conditions you found in part (f). Show work.
- (j) What is our book's definition (or the glossary) of the nullspace of a matrix  $A$ , or Null  $A$ ?
- (k) What is the augmented matrix corresponding to Null  $A$  using this matrix?
- (l) Use **prs2s22.mw** to reduce this augmented matrix to reduced row echelon form and show or write the output.
- (m) By-hand, starting from the reduced row echelon form of the augmented matrix in part (l), show by-hand work to solve for Null  $A$ .
- (n) Write a basis for Null  $A$ .
- (o) What is the geometry of Null  $A$ ? Explain.
- (p) In **prs2s22.mw**, execute the NullSpace command to see the basis that Maple provides. Show or write the output.
- (q) Compare and contrast Maple's basis for the nullspace to the basis you solved for by-hand in part (n)? Discuss.
- (r) Does the nullity-rank theorem hold? Show why or why not.

#### 4. Matrix Algebra Applications to Least Squares

Least squares analysis is very useful in many real-life applications where we want a best fit line (it's linear!) and in other applications for machine learning. Here we see connections to many class topics! We apply column space, inverse, matrix algebra, matrix multiplication, nullspace, solving linear equations, and transpose:

- (a) Inside Maple **prs2s22.mw** is real-life data reported to me last summer from a variety of people who measured their armspan and their height in  $cm$ . Let's look at the first 3 data points to highlight the matrix algebra connections to least squares.

armspan in $cm$	height in $cm$
147.95	156.21
193	185
161	162

If we want the line of best fit,  $b + xm = y$  (i.e.  $y = mx + b$  rewritten, where  $x$  is the  $x$ -coordinate of armspan in  $cm$ ,  $y$  is the  $y$ -coordinate of height in  $cm$ ,  $m$  is the slope and  $b$  is the  $y$ -intercept), and the line actually went through all 3 of the points, then we would have 3 equations:

$$b + 147.95m = 156.21 \text{ (corresponding to } (147.95, 156.21) \text{ being on the line)}$$

$$b + 193m = 185 \text{ (corresponding to } (193, 185) \text{ being on the line)}$$

$$b + 161m = 162 \text{ (corresponding to } (161, 162) \text{ being on the line)}$$

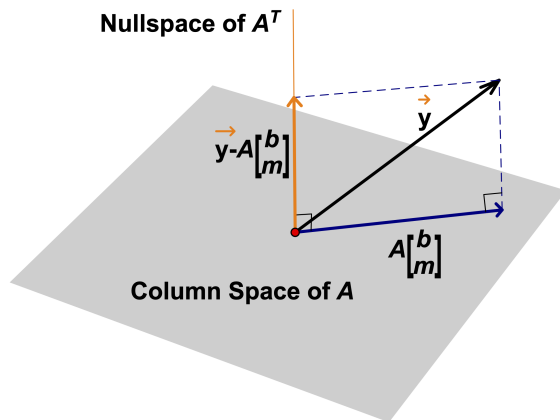
Show that this system of equations is equivalent to the matrix system  $A \begin{bmatrix} b \\ m \end{bmatrix} = \vec{y}$ , where

$$A = \begin{bmatrix} 1 & 147.95 \\ 1 & 193 \\ 1 & 161 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} 156.21 \\ 185 \\ 162 \end{bmatrix} \text{ (i.e. that } \vec{y} \text{ in the column space of } A!). \text{ Specifically,}$$

multiply  $\begin{bmatrix} 1 & 147.95 \\ 1 & 193 \\ 1 & 161 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix}$  and then set equal to  $\vec{y} = \begin{bmatrix} 156.21 \\ 185 \\ 162 \end{bmatrix}$ . Show this and compare to the 3 equations above.

- (b) Notice that the first column of 1 in  $A$  comes from 1 always as a coefficient of  $b$  in  $b + xm = y$ . However, it is very rare for the best fit line to go through all the data points, so  $A \begin{bmatrix} b \\ m \end{bmatrix} = \vec{y}$

will usually be inconsistent. Instead, we'll find a vector  $\begin{bmatrix} b \\ m \end{bmatrix}$  so that  $A \begin{bmatrix} b \\ m \end{bmatrix}$  is in the column space of  $A$  and is as close to  $\vec{y}$  as possible, as in this image showing projection by creating a right triangle, the shadow  $\vec{y}$  makes in the column space when light rays come in perpendicular to the column space:



We'll also have  $\vec{y} - A \begin{bmatrix} b \\ m \end{bmatrix}$  in the nullspace of  $A^T$  so that  $A^T(\vec{y} - A \begin{bmatrix} b \\ m \end{bmatrix}) = \vec{0}$ , which in turn is equivalent to  $A^T \vec{y} = (A^T A) \begin{bmatrix} b \\ m \end{bmatrix}$ . The cool thing is that no matter the real-life data, even though  $A$  is typically not a square matrix, so it not ever invertible,  $A^T A$  is a square matrix and when  $(A^T A)$  is invertible, we can apply the inverse and solve  $A^T \vec{y} = (A^T A) \begin{bmatrix} b \\ m \end{bmatrix}$  for  $\begin{bmatrix} b \\ m \end{bmatrix} = (A^T A)^{-1} A^T \vec{y}$ . There is a lot of matrix algebra here, including associativity, which I'll show you in the re-engage, for those who are further interested. Here though, execute the Maple commands in `prs2s22.mw` to solve for  $\begin{bmatrix} b \\ m \end{bmatrix}$  numerically. Provide the line of best fit for this

data set from Maple. Write it in the form  $y = mx + b$ . (Note that this is equivalent to what programs like Excel provide, up to rounding.)

- (c) Modify the commands in the Maple file to plot the points together with the line of best fit that you just identified. Provide your code and output.
- (d) Provide another data point—your own  $x$  =armspan and  $y$  =height in *cm* (if you have a way to measure) or find some other person’s armspan and height in *cm* (say from an internet search) where you provide the source and measurements. If you are working in a group and turning in one problem set, then provide two more data points, one for each of you.  
What are the  $(x, y)$  coordinates of the new point (or points)?
- (e) Add them to the Maple file **prs2s22.mw** by adding to  $A$  and  $\vec{y}$  and execute to use the linear algebra least squares method. Provide your code and output.
- (f) What is the new line of best fit in the form  $y = mx + b$ . Give all the decimals that Maple outputs.
- (g) Why is armspan and height interesting to look at? If you don’t already know, search to find the connection to Leonardo da Vinci.

We can apply least squares to to a wide variety of real-life settings and machine learning with multiple regressions, planes and nonlinear curves, still basically using what we did above, with a bit of extension. If you are interested in this for a final project, see 6.6!

5. Proper citations or smiley face or similar, look back and smiley face or similar, collate to one PDF:

- (a) Regardless of whether you talked to others or used other sources, be sure that your project consists of products that you and your partner (if any) create yourselves and in your own words. Give proper credit to anyone you talked to, other than if you are turning this in with a partner or talked to me, and give proper credit to any source citations. If this doesn’t apply or you already completed this above, then write a smiley face, N/A, or similar.
- (b) Look back and then write a smiley face or similar: take the time to reflect and ensure that you have answered all parts of the questions, showed work, included your by-hand and Maple work, and explained your reasoning in your own group’s words (there is no need for full sentences). A good rule of thumb in deciding how much to write is to write enough so that a classmate who hasn’t yet solved the problem could understand what you are doing and why—how you reached your conclusion from the computation or example—so they are persuaded of its validity by the logic and clarity of your reasoning. Write a smiley face or similar.
- (c) Collate your work into one PDF for submission to the ASULearn assignment. If you are turning this in with a partner, turn in one complete project writeup in **one of your ASULearn assignments**. Electronically, you can append PDFs you create from Maple to the end of your other PDFs, like by using Preview on a Mac or PDFsam on a PC. Or, if you are physically printing, you can print your Maple work and then append it to the end of handwritten work and then scan it all in to one file. If you have a phone or tablet, apps like Adobe Scan or CamScanner can work well to scan work to one full size multipage PDF. You can also use many printers or photo copiers to scan to PDFs—the school library lists that as an option and they can help:

<https://library.appstate.edu/services-search/print-zone-tech-help>.