ps4a1.mw

Dr. Sarah's Problem Set 4 Solutions

```
> with(LinearAlgebra): with(plots):
```

Warning, the name changecoords has been redefined

4.1 number 36

To see if v can be written as a linear combination of u1, u2 and u3, we set v=c1 u1+c2 u2+c3 u3

and see if we can solve this system of equations. We set up an augmented matrix to solve for c1, c2 and c3:

```
> M:=Matrix([[1,2,-3,-1],[3,-1,2,7],[5,3,-4,2]]);
```

$$M := \begin{bmatrix} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{bmatrix}$$

```
> ReducedRowEchelonForm(M);
```

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & \frac{-11}{7} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Looking at the last row, we see that the system is inconsistent since 0c1+0c2+0c3 can never equal 1. Hence there is no solution and so v cannot be written as a linear combination of u1, u2 and u3.

By Hand: To do this by hand, we use the same setup, but we row reduce the system by hand. First we perform the operations r2=-3r1+r2, and r3=-5r1+r2 and obtain the matrix

```
> Matrix([[1,2,-3,-1],[0,-7,11,10],[0,-7,11,7]]);
```

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{bmatrix}$$

Then we perform the operation r3=-r2+r3 and obtain the matrix

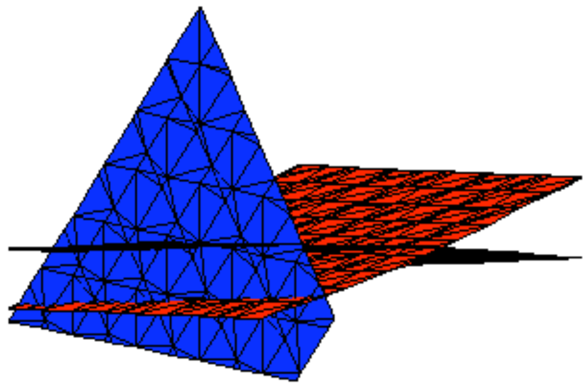
> Matrix([[1,2,-3,-1],[0,-7,11,10],[0,0,0,-3]]);

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Looking at the last row, we see that the system is inconsistent since 0c1+0c2+0c3 can never equal -3. Hence there is no solution and so v cannot be written as a linear combination of u1, u2 and u3.

Geometric Visualization: We can plot the column vectors themselves to see that v, the last column of the matrix, does not lie in the plane formed by the first three columns u1, u2, and u3 (which do happen to all lie in one plane):

We can also plot as we did in section one to see if the planes formed by the complete rows of the matrix intersect - they do not:



Maple Visualization Code for those who are interested

```
> a:=polygonplot3d([[0,0,0],[1,3,5],[3,2,8],[2,-1,3]],color=yellow):
  b:=polygonplot3d([[0,0,0],[1,3,5],[-2,5,1],[-3,2,-4]],color=yellow):
  c:=spacecurve([-3*t,2*t,-4*t,t=0..1],color=green, thickness=2):
  c2:=textplot3d([-3,2,-4,`      u_3`],color=black):
  d:=spacecurve([t,3*t,5*t,t=0..1],color=red, thickness=2):
  d2:=textplot3d([1,3,5, `      u_1`],color=black):
  e:=spacecurve([2*t,-t,3*t,t=0..1],color=blue, thickness=2):
  e2:=textplot3d([2,-1,3,`u_2      `],color=black):
  f:=spacecurve([-t,7*t,2*t,t=0..1],color=magenta, thickness=2):
  f2:=textplot3d([-1,7,2,`   v`],color=black):
  display(a,b,c,c2,d,d2,e,e2,f,f2);
```

```
> a:=implicitplot3d({x+2*y-3*z+1},x=-1/2..2,y=-2..2,z=-2..2, color=red):
  b:=implicitplot3d({3*x-y+2*z-7},x=-1/2..2,y=-2..2,z=-2..2, color=blue):
  c:=implicitplot3d({5*x+3*y-4*z-2},x=-1/2..2,y=-2..2,z=-2..2):
  display(a,b,c);
```

4.1 number 44

To see if the zero vector can be written as a nontrivial linear combination of v1, v2 and v3, we set

(0,0,0)=c1 v1+c2 v2+c3 v3 and see if we can solve this system of equations. We set up an augmented matrix to solve for c1, c2 and c3:

```
> M:=Matrix([[1,-1,0,0],[0,1,1,0],[1,2,3,0]]);
```

$$M := \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

> ReducedRowEchelonForm(M);

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Looking at the last row, we see that the system is consistent since 0c1+0c2+0c3 always equals 0. Hence because we have 3 unknowns and have reduced down to 2 consistent equations(c1+c3=0 and c2+c3=0), we know there will be an infinite number of solutions. We can take c3=t, and use the other two equations to obtain c2=-t and c1=-t. This holds for any real number t. To write the zero vector as a linear combination of the other vectors, we need to take one set of constant solutions to this system. We can take t=1, and obtain c1=-1, c2=-1 and c3=1. Hence

(0,0,0)= -v1 -v2 + v3, as desired.

By Hand: To do this by hand, we use the same setup, but we row reduce the system by hand. First we perform the operations r3=-r1+r3 and obtain the matrix

> Matrix([[1,-1,0,0],[0,1,1,0],[0,3,3,0]]);

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

Then we perform the operation r3=-3r2+r3 and obtain the matrix

> Matrix([[1,-1,0,0],[0,1,1,0],[0,0,0,0]]);

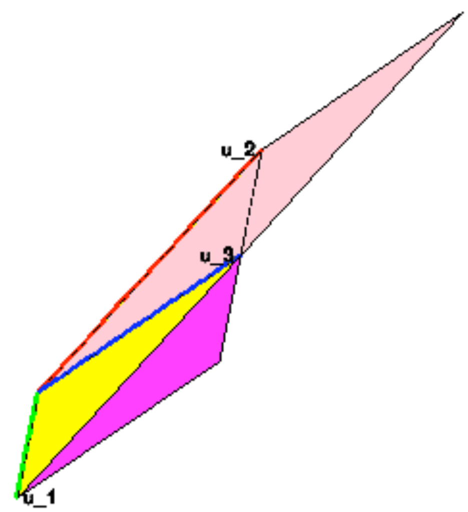
$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

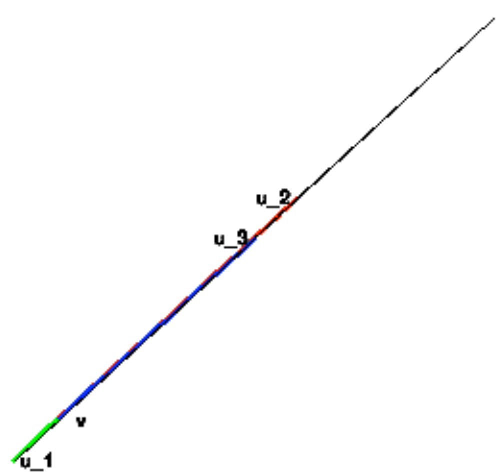
Looking at the last row, we see that the system is consistent since 0c1+0c2+0c3 always equals 0. Hence because we have 3 unknowns and have reduced down to 2 consistent equations(c1-c2=0 and c2+c3=0), we know there will be an infinite number of solutions. We can take c3=t, and use the other two equations to obtain c2=-t and c1=-t. This holds for any real number t. To write the zero vector as a linear combination of the other vectors, we need to take one set of constant solutions to this system. We can take t=1, and obtain c1=-1, c2=-1 and c3=1. Hence

(0,0,0)= -v1 -v2 + v3, as desired.

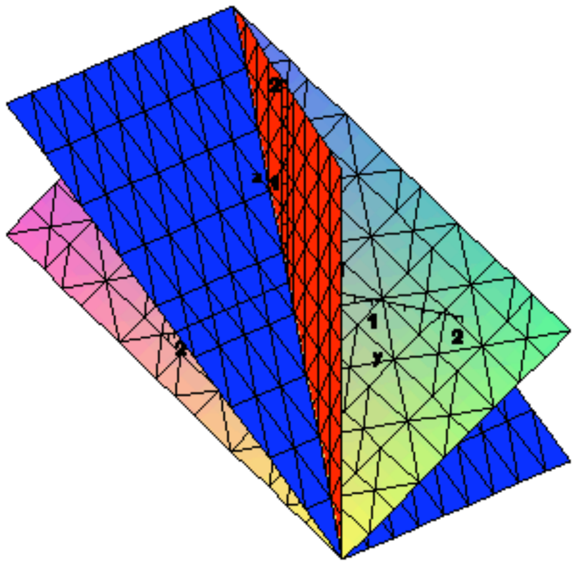
Geometric Visualization

From playing around with the picture, we can see that u1, u2, and u3 all lie in the same plane. Hence there is redundance built into this system.





We can also plot as we did in chapter 1 -- the three planes formed by the rows of the original matrix intersect in a line through the origin, and so there is a non-trivial way to write (0,0,0).



Maple Visualization Code for those who are interested

```
> a:=polygonplot3d([[0,0,0],[1,0,1],[0,1,3],[-1,1,2]],color=yellow):
  b:=polygonplot3d([[0,0,0],[1,0,1],[1,1,4],[0,1,3]],color=magenta):
  c:=polygonplot3d([[0,0,0],[-1,1,2],[-1,2,5],[0,1,3]],color=pink):
  d:=spacecurve([t,0,t,t=0..1],color=green, thickness=2):
  d2:=textplot3d([1,0,1,`       u_1`],color=black):
  e:=spacecurve([-t,t,2*t,t=0..1],color=red, thickness=2):
  e2:=textplot3d([-1,1,2, `u_2       `],color=black):
  f:=spacecurve([0,t,3*t,t=0..1],color=blue, thickness=2):
  f2:=textplot3d([0,1,3,`u_3       `],color=black):
  g:=textplot3d([0,0,0,`       v`],color=black):
  display(a,b,c,d,d2,e,e2,f,f2,g);

> a:=implicitplot3d({x-y},x=-2..2,y=-2..2,z=-2..2, color=red):
  b:=implicitplot3d({y+z},x=-2..2,y=-2..2,z=-2..2, color=blue):
  c:=implicitplot3d({x+2*y+3*z},x=-2..2,y=-2..2,z=-2..2):
  display(a,b,c);
```

VLA Cement Mixing

Concrete mix, which is used in jobs as varied as making sidewalks and building bridges, is comprised of five main materials: cement, water sand, gravel and fly ash. By varying the percentages of these materials, mixes of concrete can be produced with differing characteristics. For example, the water-to-cement ratio affects the strength of the final mix, the sand-to-gravel ratio affects the "workability" of the mix, and the fly ash-to-cement ratio affects the durability. Since different jobs require concrete with different characteristics, it is important to be able to produce custom mixes.

Assume you are the manager of a building supply company and plan to keep on hand three basic mixes of concrete from which you will formulate custom mixes for your customers. The basic mixes have the following characteristics:

	Super-Strong Type S	All-Purpose Type A	Long-life Type L
cement	20	28	12
water	10	10	10
sand	20	25	15
gravel	10	5	15
fly ash	0	2	8

Each measuring scoop of any mix weighs 60 grams, and the numbers in the table give the breakdown by grams of the components of the mix. Custom mixes are made by combining the three basic mixes. For example, a custom mix might have 10 scoops of Type S, 14 of Type A, and 7.5 of type L. We can represent any mixture by a vector in representing the amounts of cement, water, sand, gravel, and fly ash in the final mix. The basic mixes can therefore be represented by the following vectors:

```
> S := Vector([20,10,20,10,0]):A := Vector([18,10,25,5,2]):
  L := Vector([12,10,15,15,8]):
```

a) Give a practical interpretation to the linear combination: .

This represents a mix with 3 parts Super-Strong Type S, to 5 parts All-Purpose Type A, to 2 parts Long-life Type L. Since each scoop weights 60 grams, this mix will weigh 60*(3+5+2)=600 grams.

```
> 3*S+5*A+2*L;
```

$$\begin{bmatrix} 174 \\ 100 \\ 215 \\ 85 \\ 26 \end{bmatrix}$$

Notice that this mix has a high sand to gravel ratio (215:85) and so it will have a high degree of "workability". This mix has the fly ash to cement ratio of (26:174) so it has a reasonable durability. Finally, the water to cement ratio of this mix is (100:174) and so it will have some strength too.

b) What does Span$\{S,A,L\}$ represent?

The Span$\{S,A,L\}$ represents ALL linear combinations of S, A and L, so this represents all of the mixes that are combinations of these three basic mixtures. By changing the coefficients, we can obtain mixes of varying workability, durability, and strength. (You will explore more on this in the next problem set.)

c) A customer requests 6 kilograms (6000 grams) of a custom mix with the following proportions of cement, water, sand, gravel, and fly ash: 16:10:21:9:4. Find the amounts of each of the basic mixes (S, A, and L) needed to create this mix.

This problem is just like the application we did in class on coffee blends. The first step is to make sure we understand the matrix multiplication. If we write the matrix as

$$\begin{array}{c|ccc|}
 & \textit{Super-Strong} & \textit{All-Purpose} & \textit{Long-life} \\
 & \textit{Type S} & \textit{Type A} & \textit{Type L} \\
\textit{cement} & 20 & 28 & 12 \\
\textit{water} & 10 & 10 & 10 \\
\textit{sand} & 20 & 25 & 15 \\
\textit{gravel} & 10 & 5 & 15 \\
\textit{fly ash} & 0 & 2 & 8
\end{array}$$

Then it will act on a 3x1 column vector with entries a, b and c, and will output a 5x1 column vector.

So, for example, if we look at the first row acting on this a, b, c column vector, and giving us the first entry in the 5x1 column vector, let's see what this would mean.

The amount (in grams) of cement per scoop of Type S times a + the amount of cement per scoop of Type A times b + the amount of cement per scoop Type L times c

would tell us that a must be the number of scoops of Type S, b must be the number of scoops of Type A, and c must be the number of scoops of Type C, and this product gives us the amount (in grams) of cement in the mixture we create. Notice that this multiplication uniquely defines what the 3x1 and 5x1 column vectors must be.

The 3x1 column vector must be [number of scoops of Type S,
number of scoops of Type A, number of scoops of Type C]

The 5x1 column vector must be
[amount of cement, amount of water, amount of sand, amount of gravel, amount of fly ash] in the final mix.

Testing out M by Redoing Part a Using This Matrix.

What I've done above is enough to uniquely define the matrix,

let's test this out by redoing part a using this matrix.

```
> M:=Matrix([[20,18,12],[10,10,10],[20,25,15],[10,5,15],[0,2,8]]); partA:=Vector([3,5,2]);
```

$$M := \begin{bmatrix} 20 & 18 & 12 \\ 10 & 10 & 10 \\ 20 & 25 & 15 \\ 10 & 5 & 15 \\ 0 & 2 & 8 \end{bmatrix}$$

$$partA := \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

```
> M.partA;
```

$$\begin{bmatrix} 174 \\ 100 \\ 215 \\ 85 \\ 26 \end{bmatrix}$$

This is the same answer I got in part a, and it has 600 grams as it should (see part a).

Why Can't the Transpose of M be the Correct Matrix?

Why can't the transpose be used? Well the transpose would be a 3x5 matrix. It would act on a 5x1 column vector, and would output a 3x1 column vector. But notice that with the transpose matrix, the columns add up to all different numbers

(while with M itself, each column add up to 60 grams, which is the measuring scoop of any mix). Also, if I multiply the transpose by the above answer from part a

> Transpose(M).M.partA;

$$\begin{bmatrix} 9630 \\ 9984 \\ 7796 \end{bmatrix}$$

I don't get the desired 3,5,2 at all. That is because the multiplication doesn't make any sense. If I take the first row of the transpose matrix,

which is 20, 10, 20, 10, 0, and multiply by a 5x1 column vector (d,e,f,g,h)

then let's look at the common sense of this

The amount of cement (in grams) per scoop of type S times d +The amount of water per scoop of type S times e +... +The amount of fly ash per scoop of Type S times f, then the only thing that would make sense for the the 5x1 column vector would be for d, e, f, g, and h to ALL be just the number of scoops of type S. But then if we try and multiply out the second column of the matrix by this 5x1 column vector, it would only make sense if d,e,f,g, and h were ALL just the number of scoops of type A. This is a contradiction, so the multiplication does not make sense this way.

Now let's go back and answer the question (Of course your solution would not have been this detailed - I'm going through it carefully, since there seem to still be some people who are confused about how to apply matrices in real life applications such as this - you first MUST see what multiplication MAKES SENSE before you do anything else).

We want 6000 grams of a custom mix with the proportions of cement, water, sand, gravel, and fly ash: 16:10:21:9:4. Notice that there are 60 grams in this mix (by adding 16+10+21+9+4), and so if we want 6000 grams, then we must multiply the proportions by 100.

Then we want the amounts of each of the basic mixes (S, A, and L) needed to create this mix which has 1600 grams of cement, 1000 grams of water, 2100 grams of sand, 900 grams of gravel and 400 grams of fly ash. Thus, we want to solve for the 3x1 vector so that

M times the vector = the 5x1 column vector with the entries (1600,1000,2100,900,400). Since M is a 5x3 matrix, we cannot use the matrix method (nor Cramer's method) to solve, and so we MUST use the augmented matrix method.

```
> j := Vector([1600,1000,2100,900,400]);
  N := Matrix([S,A,L,j]); ReducedRowEchelonForm(N);
```

$$j := \begin{bmatrix} 1600 \\ 1000 \\ 2100 \\ 900 \\ 400 \end{bmatrix}$$

$$N := \begin{bmatrix} 20 & 18 & 12 & 1600 \\ 10 & 10 & 10 & 1000 \\ 20 & 25 & 15 & 2100 \\ 10 & 5 & 15 & 900 \\ 0 & 2 & 8 & 400 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 56 \\ 0 & 0 & 1 & 36 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, since we have 3 unknowns, and 3 consistent equations (the last two rows of 0s are consistent but trivial)

we obtain the unique solution of 8 scoops of type S, 56 scoops of type A, and 36 scoops of type L. Just as a check, notice that this gives us

100 scoops total x 60 grams/scoop = 6000 grams total, as desired.

d) Is the solution unique? Explain.

We can see from the matrix that the solution is unique. In the original system we have 3 unknowns and 5 equations and in the reduced system, we have 3 non-trivial equations (the last two rows of 0s), yielding a consistent system with exactly one solution for the coefficients. Some students misquoted the book: If a system is nxn then it has a unique solution exactly when the coefficient matrix is invertible. This statement does not tell us that non-square matrices can't have unique solutions. **This problem continues on the next problem set.**

4.2 number 22

The set of 2x2 nonsingular matrices with the standard matrix operations is not a vector space. To show this, we will show that axiom 1 is violated, ie there exists u in V, there exists v in V s.t. u+v is not in V. Take u=

```
> u:=Matrix([[1,0],[0,1]]);
```

$$u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that u is in V since the determinant of u is 1 and so u is invertible and singular (see 4.2 number 21). Take v=

> v:=Matrix([[-1,0],[0,-1]]);

$$v := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Notice that v is in V since the determinant of v is 1. Now look at u+v=

> u+v;

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that u+v is not in V since the determinant of u+v is 0, and so u+v is singular. Hence V violates axiom 1, and so it is not a vector space.

Natural Numbers

To prove that the natural numbers are not a subspace of R, we will show that number 6 in the definition of vector space does not hold, ie, there exists c in R, there exists u in natural numbers so that cu is not a natural number. Take c=pi which is real. Take u=3, which is a natural number. Notice that cu = 3pi which is not a natural number. Hence 6 is violated and so the natural numbers do not form a subspace of R.

>True or False: The line x+y=0 is a vector space.

This is a true statement. On page 198 number 2, the test says that a line that passes through the origin is a subspace of R^2. Since (0,0) satisfies x+y=0, then this is a line that passes through the origin. Since it is a subspace of R^2, by the definition of subspace, it is a vector space itself.

Subset of R^3 - Solutions to 2x-3y+4z=5

Part A: To show that the subset of R^3 consisting of all solutions to the equation 2x-3y+4z=5 is not a subspace of R^3, we will show that axiom 1 in the definition of subspace is violated, ie there exists u in the set, there exists v in set so that u+v is not in the set. Take u=(1,-1,0). Notice that u is in the subset since it satisfies the equation 2x-3y+4z=5, as 2(1)-3(-1)+4(0)=2+3+0=5. Also take v=(1,-1,0). Now u+v=(2,-2,0). To show that u+v is not in the subset, we will show that it does not satisfy 2x-3y+4z=5. Notice that 2(2)-3(-2)+4(0)= 4+6 +0 = 10, which is NOT equal to 5, as desired. Thus this subset violates number 1 in the definition of subspace and so it is not a subspace.

4.3 number 14 part D

Look at V=the set of nxn matrices, which we know is a vector space under the usual matrix operations.

We want to know if the following subset is a subspace. We'll assume that n is 2 or greater, since otherwise,

we are just looking at a 1x1 array of reals, and then some of the answers below (such as Part D) can change answers.

Part D: The set of all nxn singular matrices is not a subspace.

This set of all nxn singular matrices with the standard matrix operations is not a subspace. To show this, we will show that axiom 1 is violated, ie there exists u in the subset, there exists v in the subset s.t. u+v is not in V. Take u= the nxn matrix which has 0s everywhere except along the main diagonal, and which has a 0 on the 1,1 spot of the main diagonal, but a 1 on the j,j spot of the main diagonal everywhere except that 1,1 spot. Ie u is almost the nxn identity matrix, except it has a 0 instead of a 1 on that 1,1, spot. Notice that the first row of u is a row of 0s, and so u has determinant 0. Hence us is an nxn singular (see pages 66 and 132 Thoerem 3.7) matrix and so u is in our subset. Take v = nxn matrix which has 0s everywhere except is has a 1 in the 1,1, spot. Notice that the determinant of v is also 0, since row 2 is a row of all 0s. Yet, u+v is the nxn identity matrix, which has determinant 1, and so is non-singular. Hence u+v is not in our subset, and so this set is not a subspace since it violates axiom 1.

Extra Credit

To show that the subset of R^5 consisting of all the solutions of the nonhomogenous equation Ax=b, where A is a given 4x5 matrix and b is a given non-zero vector in R^4 is not a subspace, we must look at cases.

Dr. Sarah's comments: Note that we cannot use the inverse since the matrix is not square. Also, we cannot use p. 134 Equivalent conditions for a non-singular matrix which

says that Ax=b has a unique solution for every nx1 matrix b, since this ONLY APPLIES to square (nxn) matrices. We do know that A acts on a 5x1 matrix and is supposed to output a 4x1 matrix that matches b, which is non-zero. Hence, we know that we have 5 unknowns and 4 equations.

Recall that in the case with 3 unknowns and 2 equations

(ie 3 unknowns in R^3 would be the equation of a plane), there can be no solutions (parallel planes) or infinitely many solutions (2 intersecting planes have a line of intersection, if the equation is the same - ie just the same plane - then we would have the whole plane as the intersection.).

Back to our proof...

Case 1: If Ax=b has NO solutions, then this subset is empty, and so it violates the condition that we have a non-empty subset. Hence the subset is not a subspace.

Case 2: There is a solution to the system (I don't care whether there is a unique solution or whether there are infinitely many solutions, although I could probably prove that there must be infinitely many solutions if b is not zero - if b is zero, there can be 1 solution or infinitely many). We will show that axiom number 2 in the definition of subspace is

violated, ie there exists c in R, there exists x in subset so that cx is not in the subset. Take c=0. Notice that c is real so it is in R.

Take x as any solution to the system Ax=b. We know there must be at least one since we are in case 2. Now look at cx=0x, which is the 5x1 column vector with all zero entries.

Yet then A(cx)=A(0x) equals the 4x1 zero vector, since cx is the 5x1 zero vector. But we know that b is not the zero vector since our system is nonhomogeneous,

and so A(cx) is not equal to b. Therefore cx is not a solution to our system, and so the subset violates axiom 2 in the definition of subspace. Hence the subset is not a subspace.

In either case, the subset is not a subspace, as desired.