ps5a.mw

Dr. Sarah's Problem Set 5 Solutions

> with(LinearAlgebra): with(plots):

Warning, the name changecoords has been redefined

4.4 number 16

Determine whether S={(1,0,3),(2,0,-1),(4.0,5),(2,0,6)} spans R^3. We want to know whether any vector (u1,u2,u3) in R^3 can be written can be written as a linear combination of the vectors in S. So, we can set up the system

M *column vector(c1,c2,c3,c4)=column vector(u1,u2,u3) to see if we always get a solution. Since M is a 3x4 matrix, we cannot use the inverse method to solve, but we could form the augmented matrix. Since all of these vectors have 0 as a second coordinate, I will show that (0,1,0) cannot be written as a linear combination of the vectors in S. Here is the augmented matrix:

> M:=Matrix([[1,2,4,2,0],[0,0,0,0,1],[3,-1,5,6,0]]);

$$M := \begin{bmatrix} 1 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & -1 & 5 & 6 & 0 \end{bmatrix}$$

> ReducedRowEchelonForm(M);

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the last row gives us 0c1+0c2+0c3+0c4=1, we know that the system is inconsistent, and so (0,1,0) cannot be written as a linear combination of the vectors in S. Hence S does not span R^3.

In fact, the third and fourth vectors can be written as linear combinations of the first two:

(4,0,5)=2(1,0,3)+(2,0,-1)

(2,0,6)=2(1,0,3)

but the first two vectors are not multiples of each other, so the first two vectors are linearly independent. Hence this set spans a plane in R^3 (the x-z plane), and {(1,0,3),(2,0,-1)} form a basis for that plane, but the entire set does not span all of R^3 (because everything cannot be represented using them), nor does the entire set yield linearly independent vectors in R^3 (because of redundancy in the 4 vectors). Notice below that if I set up the general spanning system and reduce, I obtain:

```
> N:=Matrix([[1, 2, 4, 2,u1], [0, 0, 0, 0,u2], [3, -1, 5, 6,u3]]);
```

$$N := \begin{bmatrix} 1 & 2 & 4 & 2 & u1 \\ 0 & 0 & 0 & 0 & u2 \\ 3 & -1 & 5 & 6 & u3 \end{bmatrix}$$

```
> ReducedRowEchelonForm(N);
```

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

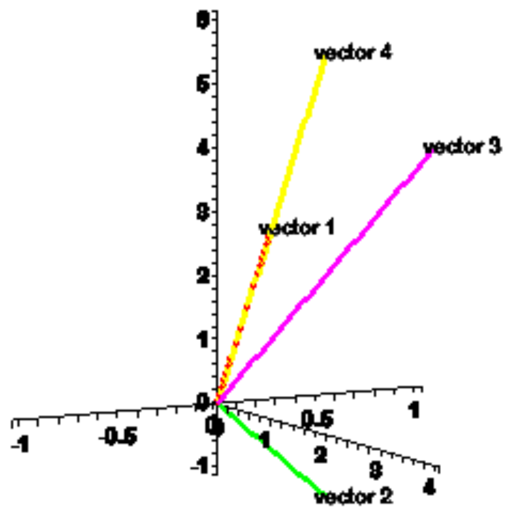
```
> GaussianElimination(N);
```

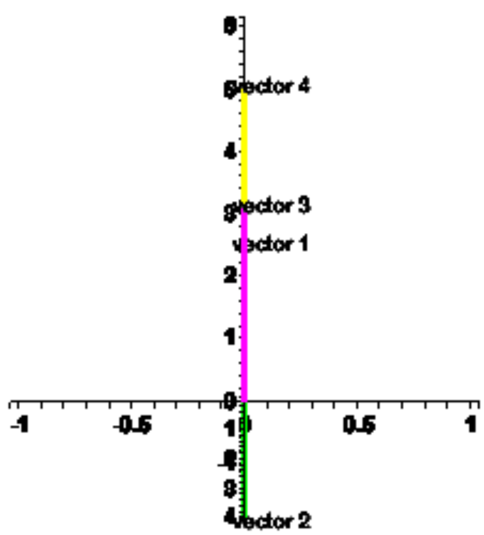
$$\begin{bmatrix} 1 & 2 & 4 & 2 & u1 \\ 0 & -7 & -7 & 0 & u3 - 3\,u1 \\ 0 & 0 & 0 & 0 & u2 \end{bmatrix}$$

We've seen before that we have to be careful when we have unknowns - the reduced form is misleading, because we do have solutions when u2=0. The Gaussian form is a bit better in this regard - here we see that we have no variables equaling u2.

Visualization We cannot plot the rows as we did in Chapter 1, because there is no constant on the right hand side to determine the hyperplanes, and besides, they would be in 4-space (4 variables as the columns). We can visualize the column vectors in R^3: Notice that in the second picture, I have rotated it to see that all the vectors lie in the same plane.

```
> a1:=spacecurve({[t,0,3*t,t=0..1]},color=red, thickness=2):
  a2:=textplot3d([1,0,3,`       vector 1`],color=black):
  b1:=spacecurve({[2*t,0,-t,t=0..1]},color=green, thickness=2):
  b2:=textplot3d([2,0,-1,`       vector 2`],color=black):
  c1:=spacecurve({[4*t,0,5*t,t=0..1]},color=magenta, thickness=2):
  c2:=textplot3d([4,0,5,`       vector 3`],color=black):
  d1:=spacecurve({[2*t,0,6*t,t=0..1]},color=yellow, thickness=2):
  d2:=textplot3d([2,0,6,`       vector 4`],color=black):
  display(a1,a2, b1,b2,c1,c2,d1,d2);
```





4.5 number 24

To determine whether S={(1,5,3),(0,1,2),(0,0,6)} is a basis for R^3, we must determine if the vectors in S are linearly independent and span the set. First, we set up the linear independence equation M*column vector (c1,c2,c3)=(0,0,0), where M is

> M:=Matrix([[1,0,0],[5,1,0],[3,2,6]]);

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}$$

> Determinant(M);

$$6$$

Since the determinant of M is non-zero, then we know that the system has only the trivial solution, and so the set is linearly independent. Since any basis for R^3 consists of 3 vectors, and we have the set S consisting of 3 linearly independent vectors, then we know that S spans also and is a basis for R^3 by Theorem 4.12.

4.5 number 48

In order to find a basis for R^3 that includes the set S={(1,0,2),(0,1,1)}, we know we need to add exactly one vector v so that the set remains linearly independent via Theorem 4.12 and the fact that R^3 has dimension 3. So, we just need to add one vector that is not in

Span(S)=plane formed by those two vectors. For example, take v=(1,0,0) [Like in class when we worked on the group problems, I used the rows to come up with v - ie if I made c1=1 and c2=0, then I looked at row 3 to come up with a line that would lead to an inconsistent system for v being written as a linear combination of the other two vectors - I could have taken anything other than 2 in that last spot]. To show that these set of three vectors is linearly independent,

we look at the system

M*column vector (c1,c2,c3)=(0,0,0), where M is

> M:=Matrix([[1,0,1],[0,1,0],[2,1,0]]);

$$M := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

> Determinant(M);

$$-2$$

Since the determinant of M is not zero, we know that the homogeneous system with this as a coefficient matrix has only the trivial **0** vector solution, and so the set we have formed is linearly independent, and hence is a basis for R^3.

Cement Mixing Continued

This problem follows up on the problem from the last problem set. Below is the table that describes the composition of the three basic mixtures of concrete, S, A, L. Consider two custom mixtures, X and Y, below. (All five vectors have been entered below.)

	Super-Strong Type S	All-Purpose Type A	Long-life Type L
cement	20	18	12
water	10	10	10
sand	20	25	15
gravel	10	5	15
fly ash	0	2	8

	Type X	Type Y
cement	12	15
water	12	10
sand	12	20
gravel	12	10
fly ash	12	5

```
> S := Vector([20,10,20,10,0]):

> A := Vector([18,10,25,5,2]):

> L := Vector([12,10,15,15,8]):

> X := Vector([12,12,12,12,12]):

> Y := Vector([15,10,20,10,5]):

> zero:= Vector([0,0,0,0,0]):
```

(a) Show that {S, A, L} is a linearly independent set of vectors. What practical advantage does that have?

To show that {S, A, L} is a linearly independent set of vectors, we follow along with p. 192 in the book. We write the homogenous system of equations c1S +c2A +c3L = **0**, and use Gauss-Jordan on the augmented matrix

```
> LISAL := Matrix([S,A,L,zero]);
  ReducedRowEchelonForm(LISAL);
```

$$LISAL := \begin{bmatrix} 20 & 18 & 12 & 0 \\ 10 & 10 & 10 & 0 \\ 20 & 25 & 15 & 0 \\ 10 & 5 & 15 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that this system has only the trivial solution c1=0, c2=0 and c3=0, and so {S,A,L} IS linearly independent. The practical advantage of this is the following. We know that span{S,A,L} is a vector space (it is a subspace of R^5 also) and so by p. 201 Theorem 4.9, we would know that every mix that can be obtained as a linear combination of S, A, and L, can be written in one and only one way as this linear combination.

(b) Show that we can make the custom mix Y but not the custom mix X from the three basic mixes S, A, L. What does this say about the linear independence or dependence of the sets {S, A, L, X} and {S, A, L, Y}?

Refer to my solutions for PS 4, for the setup of the matrix. As there, (see there for the explanation) to see how to use S,A and L to make a custom mix, we set up the system of equations, and use the augmented matrix

```
> SALY := Matrix([S,A,L,Y]);
  ReducedRowEchelonForm(SALY);
```

$$SALY := \begin{bmatrix} 20 & 18 & 12 & 15 \\ 10 & 10 & 10 & 10 \\ 20 & 25 & 15 & 20 \\ 10 & 5 & 15 & 10 \\ 0 & 2 & 8 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we see that Y can be written as 0S +1/2A +1/2L. Hence Y is a linear combination of S A and L, and so by p. 194 Theorem 4.8, {S, A, L, Y} is linearly dependent.

```
> SALX := Matrix([S,A,L,X]);
  ReducedRowEchelonForm(SALX);
```

$$SALX := \begin{bmatrix} 20 & 18 & 12 & 12 \\ 10 & 10 & 10 & 12 \\ 20 & 25 & 15 & 12 \\ 10 & 5 & 15 & 12 \\ 0 & 2 & 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we see that X can not be written as a linear combination of S, A and L, since the 4th row gives us 0a +0b+0c=1, which is not possible. We cannot tell whether {S, A, L, X} is linearly dependent or independent. We do know that X cannot be written as a linear combination of S, A and L, and that {S, A, L} is linearly independent, but this does not tell us whether or not one of {S, A, L} can be written in terms of the others and X. To check this, we would form the homogeneous system

```
> SALX2:=Matrix([S,A,L,X,zero]); ReducedRowEchelonForm(SALX2);
```

$$SALX2 := \begin{bmatrix} 20 & 18 & 12 & 12 & 0 \\ 10 & 10 & 10 & 12 & 0 \\ 20 & 25 & 15 & 12 & 0 \\ 10 & 5 & 15 & 12 & 0 \\ 0 & 2 & 8 & 12 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that the only solution is c1=c2=c3=c4=0, and so {S,A,L,X} is a linearly independent set of vectors.

(c) Explain why any combination of S, A, L and Y can be also achieved by a combination of just S, A and L. For example, show how to make the custom mix $3S+4A+2L+3Y$ using only S, A and L.

Recall from part b0 that Y can be written as 0S +1/2A +1/2L, so to make the custom mix $3S+4A+2L+3Y$

we substitute Y=1/2A+1/2L into

3S +4A +2L +3Y = 3S +4A +2L +3(1/2A+1/2L) = 3S +11/2 A +7/2 L.

In general, to make any mix involving Y, just substitute in Y=1/2A+1/2L in order to see that any combination of S, A, L and Y can be also achieved by a combination of just S, A and L.

(d) Define a fifth basic mix Z to add to {S, A, L, X} such that any custom mixture can be expressed as a linear combination of the set of mixes { S, A, L, X, Z }.

From the ps 4 solutions, we know that (this was the in the statement of problem 6 in Module 2)

We can represent any mixture by a vector $[c, w, s, g, f]$ in R^5 representing the amounts of cement, water, sand, gravel, and fly ash in the final mix.

Since R^5 has dimension 5, we know that it can be represented by a basis consisting of 5 vectors. We want to find Z so that {S, A, L, X, Z} is linearly independent. Since R^5 has the basis (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1) then we know that R^5 has dimension 5. Hence we can use Theorem 4.12 from p. 206 (number 1) to say that if we find Z so that {S, A, L, X, Z} is linearly independent, then this set will also be a basis for R^5 and then every mix will be written in one and only one way as a linear combination of these mixes.

Take Z as the column vector with 0 grams of everything but fly ash, and 60 grams of fly ash. (We need every mix to have 60 grams, so that is why the last entry is 60).

```
> Z:=Vector([0,0,0,0,60]):
```

Now we set up the homogeneous system of equations to test linear independence:

```
> SALXZ:=Matrix([S,A,L,X,Z,zero]); ReducedRowEchelonForm(SALXZ);
```

$$SALXZ := \begin{bmatrix} 20 & 18 & 12 & 12 & 0 & 0 \\ 10 & 10 & 10 & 12 & 0 & 0 \\ 20 & 25 & 15 & 12 & 0 & 0 \\ 10 & 5 & 15 & 12 & 0 & 0 \\ 0 & 2 & 8 & 12 & 60 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence we see that this has only the trivial solution of all 0s, and so this set is linearly indpendent, as desired.

Another way of thinking about this is as follows. Once we add Z, we will get a 5x5 matrix, and so we would just need a matrix that

has non-zero determinant, since this will have an inverse, and ensure a unique solution for each system (including the homogenous system). We see that the determinant is non-zero below:

> Determinant(Matrix([S,A,L,X,Z]));

$$360000$$

(e) Why will there still be mixes that cannot be physically produced from this set of five basic mixes? (Hint: Consider the signs of the scalar weights.) Give an example of such a mix.

Notice that from above, each mix in R^5 can be uniquely expressed as a combination of S, A, L, X, and Z.

Yet, physically, this is not always possible! we must need negative numbers times mixes, which will not be physically possible. For example, let's say that we want a custom mixture made out of S, A, L, X and Z in the ratio of cement to water to sand to gravel to fly ash of 1:2:3:4:5. Then we set up the augmented system and solve for the scoops of S, A, L, X, and Z (see ps 4 solutions):

> SALXZno:=Matrix([S,A,L,X,Z,Vector([1,2,3,4,5])]); ReducedRowEchelonForm(SALXZno);

$$SALXZno := \begin{bmatrix} 20 & 18 & 12 & 12 & 0 & 1 \\ 10 & 10 & 10 & 12 & 0 & 2 \\ 20 & 25 & 15 & 12 & 0 & 3 \\ 10 & 5 & 15 & 12 & 0 & 4 \\ 0 & 2 & 8 & 12 & 60 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{-13}{50} \\ 0 & 1 & 0 & 0 & 0 & \frac{2}{25} \\ 0 & 0 & 1 & 0 & 0 & \frac{12}{25} \\ 0 & 0 & 0 & 1 & 0 & \frac{-1}{12} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{30} \end{bmatrix}$$

Notice that we must take -1/12 of Type X, which is physically impossible since we can't have a negative quantity physically "removed" from the mix. Since the representation is unique (see part d), this tells us that we can't make the custom mix in the ratio of cement to water to sand to gravel to fly ash of 1:2:3:4:5.

Another way of doing this problem would have been to take a (5x1) column vector consiting of number of scoops of S,A,L,X,Z respectively, where one of the entries was negative, and then

multiply the 5x5 matrix with columns S, A, L, X, Z times the 5x1. This would have given us a mix that is not physically obtainable for the same reasons.

4.6 number 24

First we need to solve the system Ax=0. We will look at the augmented matrix for the system:

> M:=Matrix([[3,3,15,11,0],[1,-3,1,1,0],[2,3,11,8,0]]);

$$M := \begin{bmatrix} 3 & 3 & 15 & 11 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 2 & 3 & 11 & 8 & 0 \end{bmatrix}$$

> ReducedRowEchelonForm(M);

$$\begin{bmatrix} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that this system has an infinite number of solutions since it has four unknowns and two non-trivial solutions (and the last row is consistent but trivial). We can take x4=t, x3=s, x2=-s-2/3t, and x1=-4s-3t, where s and t are any real numbers. Hence this system yields solutions of the form (-4s-3t,-s-2/3t,s,t) = (-4s,-s,s,0)+(-3t,-2/3t,0,t) = s*(-4,-1,1,0) +t*(-3,-2/3,0,1),

where s and t are any real numbers. Hence the dimension for the solution space is 2 and a basis is {(-4,-1,1,0), (-3,-2/3,0,1)} The solutions span a plane in R^4, and if we could graph the augmented matrix rows, we would see them intersecting in this plane through the origin.

4.6 number 27

Part A: The system Ax=b is consistent since the augmented matrix

> M:=Matrix([[1,3,10,18],[-2,7,32,29],[-1,3,14,12],[1,1,2,8]]);

$$M := \begin{bmatrix} 1 & 3 & 10 & 18 \\ -2 & 7 & 32 & 29 \\ -1 & 3 & 14 & 12 \\ 1 & 1 & 2 & 8 \end{bmatrix}$$

reduces to

> ReducedRowEchelonForm(M);

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives us infinitely many solutions $z=t$, $y=5-4t$, $x=3+2t$.

Part B: The solutions of $Ax=b$ are of the form $(3+2t,5-4t,t)$ where t is any real number. Hence

$x=t(2,-4,1) +(3,5,0)$, where

x_h is $t(2,-4,1)$ and

x_p is $(3,5,0)$

If we graphed the rows of the augmented matrix, we would see them intersecting in a line. This line is not a vector space because it does not run through the origin.