1.7 Linear Independence

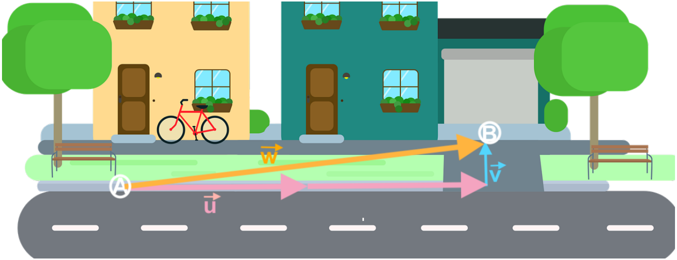
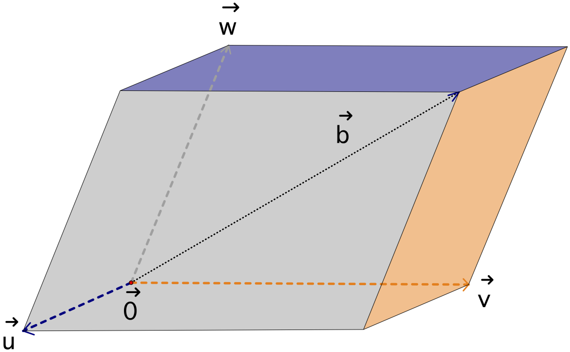


Image Credit: Vicky Klima

- $\vec{w}$ is in some sense extra: $\vec{w} = 2\vec{u} + \vec{v}$
- Informally, a set of vectors is *linearly independent* if it contains only independent directions with no redundancy
- $\vec{0}$ can't go anywhere but stay at the origin

Geometric Perspectives for Linearly Independent

- for a line only need 1 independent direction to span
- for a plane only need 2 independent directions to span
- for a volume only need 3 independent directions to span



$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

	linear combination of others?	
Is $\vec{u}$ redundant?	$\vec{u} = c_1\vec{v} + c_2\vec{w}$	
Is $\vec{v}$ redundant?	$\vec{v} = c_1\vec{u} + c_2\vec{w}$	
Is $\vec{w}$ redundant?	$\vec{w} = c_1\vec{u} + c_2\vec{v}$	

Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

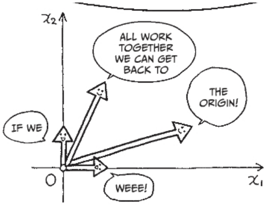
	linear combination of others?	homogeneous system
Is $\vec{u}$ redundant?	$\vec{u} = c_1\vec{v} + c_2\vec{w}$	$\vec{0} = -1\vec{u} + c_1\vec{v} + c_2\vec{w}$
Is $\vec{v}$ redundant	$\vec{v} = c_1\vec{u} + c_2\vec{w}$	$\vec{0} = c_1\vec{u} - 1\vec{v} + c_2\vec{w}$
Is $\vec{w}$ redundant?	$\vec{w} = c_1\vec{u} + c_2\vec{v}$	$\vec{0} = c_1\vec{u} + c_2\vec{v} - 1\vec{w}$

Can we find a *non-trivial* solution to $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$? If so, not linearly independent—linearly dependent!

Linear Independent Definition

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is *linearly independent* if
$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ has only the trivial solution

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ for the weights.



LINEAR DEPENDENCE

The Manga Guide to Linear Algebra by Iroha Inoue and Shin Takahashi

My Favorite Example: Setup

Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ linearly independent?

My Favorite Example: Setup

Is the set $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \right\}$ linearly independent?

Is the *only way* to write $\vec{0}$ as a linear combination of these vectors the trivial one (all weights 0) ?

$$c_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + c_2 \begin{bmatrix}4\\5\\6\end{bmatrix} + c_3 \begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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Does the homogeneous system have *only* the trivial solution?

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

My Favorite Example: Algebraically

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix} \end{array} \xrightarrow{\text{Gaussian}} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} \boxed{1} & 4 & 7 & 0 \\ 0 & \boxed{-3} & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

My Favorite Example: Algebraically

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$$\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & \end{array} \\ \left[\begin{array}{cccc} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \end{array} \xrightarrow{\textit{Gaussian}} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & \end{array} \\ \left[\begin{array}{cccc} \boxed{1} & 4 & 7 & 0 \\ 0 & \boxed{-3} & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

- The set of vectors is *not* linearly independent.
- consistent and missing a pivot for c_3 so free parameter t
  row 2: $-3c_2 - 6c_3 = 0$ so $c_2 = -2t$
  row 1: $c_1 + 4c_2 + 7c_3 = 0$ so $c_1 = -4(-2t) - 7(t) = t$
  $$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Is the *only way* to write $\vec{0}$ as a linear combination of these vectors the trivial one (all weights 0) ?

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Does the homogeneous system below have *only* the trivial solution?

No

$$\begin{bmatrix} 3 & 5 & 7 \\ -3 & -2 & -1 \\ 6 & 1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution set : $\left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t, \quad t \text{ a real number} \right\}$

Is the *only way* to write $\vec{0}$ as a linear combination of these vectors the trivial one (all weights 0) ?

$$\boxed{c_1}\begin{bmatrix}1\\2\\3\end{bmatrix}+\boxed{c_2}\begin{bmatrix}4\\5\\6\end{bmatrix}+\boxed{c_3}\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

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$$\begin{bmatrix}3&5&7\\-3&-2&-1\\6&1&-4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solution set : $\left\{\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}t, \quad t \text{ a real number}\right\}$

When $t=1$: $\boxed{1}\begin{bmatrix}1\\2\\3\end{bmatrix}+\boxed{-2}\begin{bmatrix}4\\5\\6\end{bmatrix}+\boxed{1}\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

When $t=1.5$: $\boxed{1.5}\begin{bmatrix}1\\2\\3\end{bmatrix}+\boxed{-3}\begin{bmatrix}4\\5\\6\end{bmatrix}+\boxed{1.5}\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\leftarrow$ Linear Dependance Relations

My Favorite Example: Geometrically

Is the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ linearly independent?

A set of vectors are linearly independent if they provide independent directions, i.e. for 3 vectors a volume.

Maple!

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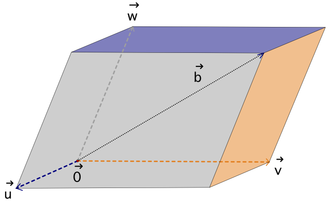
Maple!

all 3 in a plane, but only need 2 vectors to efficiently span a plane without redundancy



Which h make the columns of A l.i.?

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 2 & -8 & 6 \\ 5 & 0 & -5 & 0 \\ 7 & -4 & -3 & h \end{bmatrix}$$

Which h make the columns of A l.i.?

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 2 & -8 & 6 \\ 5 & 0 & -5 & 0 \\ 7 & -4 & -3 & h \end{bmatrix}$$

Vector equation: which h have only the trivial solution?

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 5 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 2 \\ 0 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -8 \\ -5 \\ -3 \end{bmatrix} + c_4 \begin{bmatrix} -1 \\ 6 \\ 0 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix equation: which h have a unique solution?

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 2 & -8 & 6 \\ 5 & 0 & -5 & 0 \\ 7 & -4 & -3 & h \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

l.i. by inspection and as many l.i. columns as possible

Consider $A = \begin{bmatrix} 0 & 0 & 6 & 6 & 19 & 11 \\ 3 & 12 & 9 & 3 & 26 & 31 \\ 1 & 4 & 3 & 1 & 10 & 9 \end{bmatrix}$.

(a) How do we know, by inspection, that the columns of A are not linearly independent?

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(b) give an example of a linearly independent set containing as many of the columns of A as possible.

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$$\begin{bmatrix} 0 & 0 & 6 & 6 & 19 & 11 \\ 3 & 12 & 9 & 3 & 26 & 31 \\ 1 & 4 & 3 & 1 & 10 & 9 \end{bmatrix} \xrightarrow{\text{ReducedRowEchelonForm}} \begin{bmatrix} 1 & 4 & 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

l.i. by inspection and as many l.i. columns as possible

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The set $S = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 19 \\ 26 \\ 10 \end{bmatrix} \right\}$ is an l.i. set from the columns of A.

Connections Theorem for Spanning the Entire Space

The following statements are logically equivalent.

a. $c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{b}$ is always consistent (each $\vec{b}$ is a linear combination)

b. $\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ is always consistent

c. $\left[\begin{array}{ccc|c} & & & b_1 \\ \vec{v}_1 & \ldots & \vec{v}_n & \vdots \\ & & & b_n \end{array}\right]$ reduces to a matrix with a pivot position in every row in the coefficient matrix

Connections Theorem for Linear Independence (l.i.)

The following statements are logically equivalent.

a. $c_1\vec{v}_1 + ... + c_n\vec{v}_n = \vec{0}$ has only the trivial solution $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (i.e. the def of $\{v_1, ..., v_n\}$ l.i.)

b. $\begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ has only $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

c. $\left[\begin{array}{ccc|c} & & & 0 \\ \vec{v}_1 & \dots & \vec{v}_n & \vdots \\ & & & 0 \end{array}\right]$ reduces to a matrix with a pivot position in every

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c. $\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix}$ reduces to a matrix with a pivot position in every column except the = column

Applying the Connection Theorems

Let $A_{2\times 3} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & h & 2 \end{bmatrix}$

linear independence

$$\begin{bmatrix} 3 & 1 & 4 & 0 \\ 1 & h & 2 & 0 \end{bmatrix} \xrightarrow{r_2' = -\frac{1}{3}r_1 + r_2} \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & -\frac{1}{3} + h & -\frac{4}{3} + 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & -\frac{1}{3} + h & \frac{2}{3} & 0 \end{bmatrix}$$

span

$$\begin{bmatrix} 3 & 1 & 4 & b_1 \\ 1 & h & 2 & b_2 \end{bmatrix} \xrightarrow{r_2' = -\frac{1}{3}r_1 + r_2} \begin{bmatrix} 3 & 1 & 4 & b_1 \\ 0 & -\frac{1}{3} + h & \frac{2}{3} & -\frac{1}{3}b_1 + b_2 \end{bmatrix}$$

geometry

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span

$$\begin{bmatrix} 3 & 1 & 4 & b_1 \\ 1 & h & 2 & b_2 \end{bmatrix} \xrightarrow{r_2' = -\frac{1}{3}r_1 + r_2} \begin{bmatrix} 3 & 1 & 4 & b_1 \\ 0 & -\frac{1}{3} + h & \frac{2}{3} & -\frac{1}{3}b_1 + b_2 \end{bmatrix}$$

geometry

not linearly independent as we only need 2 independent directions to span the entire $\mathbb{R}^2$ plane

span all of $\mathbb{R}^2$ as long as they aren't all on the same line

Efficiency vs Redundancy

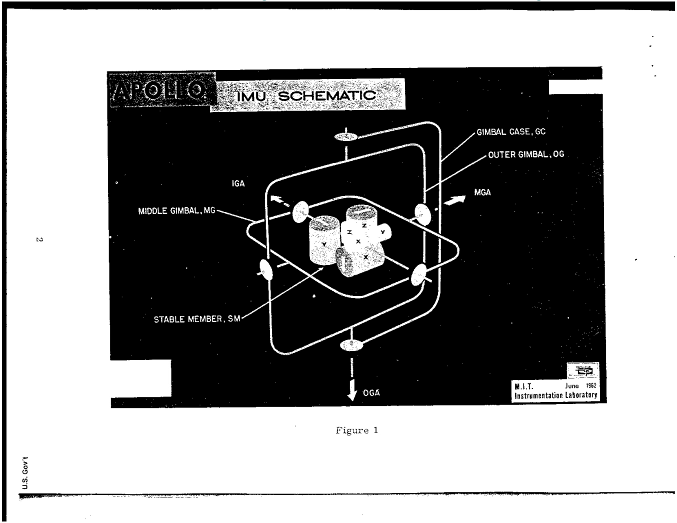


Figure 1

https://www.hq.nasa.gov/alsj/imu-2.jpg

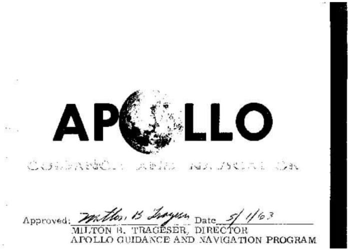
Apollo Guidance and Navigation Considerations of Apollo IMU Gimbal Lock

MIT Instrumentation Laboratory Document E-1344

David Hoag

April 1963

This report was scanned and formated by Marv Hein.


APOLLO

GUIDANCE AND NAVIGATION

Approved: Milton B. Trageser Date 5/1/63
MILTON B. TRAGESER, DIRECTOR
APOLLO GUIDANCE AND NAVIGATION PROGRAM

https://www.hq.nasa.gov/alsj/e-1344.htm