Catalog description: A study of vectors, matrices and linear transformations, principally in two and three dimensions, including treatments of systems of linear equations, determinants, and eigenvalues. Prerequisite: MAT 1120 or permission of the instructor.

Course Goals

- Develop algebraic skills
- Develop mathematical reasoning and problem solving
- Develop spatial visualization skills
- Learn about some applications of linear algebra
- An introduction to a computer algebra software system as it applies to linear algebra

Mapping of the Topics in the Catalog Description to the Text

Systems of Linear Equations: 1.1, 1.2, 1.5
Vectors: 1.3, 1.4, 1.7, 6.1
Matrices: earlier +2.1, 2.2, 2.3
Linear transformations 1.8, 1.9 (1-1 and onto eliminated), 2.7
Determinants: 3.1, 3.2, 3.3
Eigenvalues 2.8, 5.1, 5.2, 5.6

Test 2: cumulative + what we covered in 1.8, 1.9, 2.7, 2.8, 3.1, 3.2, 3.3, 5.1, 5.6, 6.1 & apps

- Formatting same as prior tests. Test 2 is majority new material.
- hw, problem sets, & clicker questions [solutions online]
- computations, definitions, critical reasoning & "big picture"
- 1.1, 1.2 & 1.5: Gaussian elimination, algebra and geometry of solutions of systems of equations...
- 1.4: connects everything together
- 1.3 and 1.7: algebra and geometry of vectors (linear combinations/mixing, span, li...)
- 2.1 and 2.2: matrix algebra: $A + B$, cA, A^T, AB, $A^{-1}_{2\times 2}$, $\det(A_{2\times 2})$
- 2.3: theorem 8: what makes a matrix invertible [connects 2.2 to 1.1, 1.2, 1.3 and 1.7] & condition #
- 1.8 (62, 65, 67-68), 1.9 (70-75), 2.7: linear transformations
- apps: Hill cipher, computer graphics/animations, computer speed and reliability
- 3.1-3.3: algebra and geometry of determinants, invertibility
- 2.8: subspace, basis, column space and null space
- 5.1: eigenvectors & eigenvalues alg & geom, nullspace$(A - \lambda I)$
- 5.6: eigenvector decomposition, limit, trajectory & populations
- 6.1 : length & angle of a vector, orthogonal vectors

Rotation: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ Dilation: $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ Horizontal Shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Projections: y=x line: $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ y-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Reflections: y=x line: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Translation: $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix}$ Others: $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

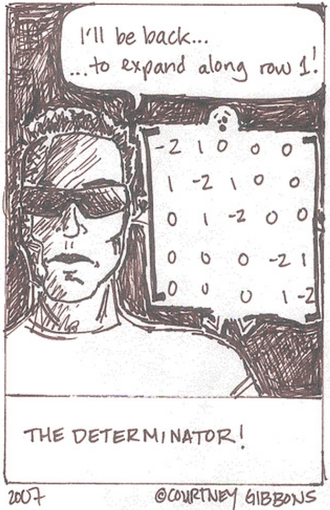
Rotate a Figure about the point $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & \dots & x_p \\ y_1 & \dots & y_p \\ 1 & \dots & 1 \end{bmatrix}$$

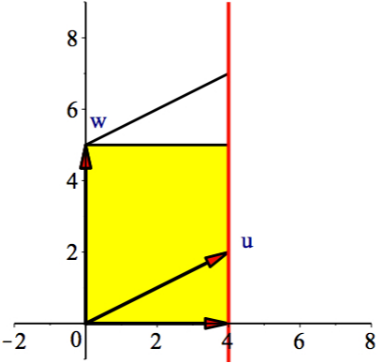
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \text{ with columns: } \vec{u} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \vec{v} = \begin{bmatrix} \text{-}\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Length (or norm) of $\vec{u} = ||\vec{u}|| \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}}$

- Useful in keeping a car on a racetrack. If we don't normalize the vectors, the car size won't be preserved.
- Here inner product $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$ induces metric on the space $||\vec{u} - \vec{v}||$ is distance between vectors as in 1.3
- Generalized inner products for nonlinear/non-Euclidean satisfy axiomatic properties like distributivity, pulling out scalars, positive definite condition
- Two vectors are orthogonal if right angle between them. One formulation of the dot product $\vec{u} \cdot \vec{v}$ is $||\vec{u}||||\vec{v}||cos\theta$, where θ is the angle between them, so the dot product is 0 exactly when the angle is $\frac{\pi}{2}$



$$-2\cdot(-1)^{1+1}\begin{vmatrix}-2 & 1 & 0 & 0\\ 1 & -2 & 0 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 1 & -2\end{vmatrix}+1\cdot(-1)^{1+2}\begin{vmatrix}1 & 1 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 1 & -2\end{vmatrix}+0s$$



Replacement $r_j' = -3r_1 + r_j$ are shear matrices when written in elementary matrix form like $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$—preserve the determinant (area, volume...) of a matrix and turns parallelograms and parallelopipeds formed by the vectors into rectangles and rectangular prisms

What Makes You Invertible

In verse

Music by One Direction & idea adapted from Art Benjamin
Interpreted by Dr. Sarah and Joel Landsberg

Baby you'll light up if one of these facts is so,
but you'll need n square columns and rows:

- Like when $\mathbb{R}^n$ is the span of the matrix columns
- That's when you know oh-oh invertible!
- If always you uniquely solve $A\vec{x}$ is $\vec{b}$
- Or if your columns have no linear dependency
- Or if matrix reduces to identity

Not zero - no no
That makes it not invertible!

Shout out if one of these facts is so...
but you'll need n square columns and rows:

- Like when your matrix determinant's non-zero

Is when you know oh-oh—that makes it invertible!

Subspace: it's linear (linear combinations)
Subspace basis: span (full row pivots) + l.i. (full column pivots)

reduced $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

column space of A=span of columns $[c_1 \vec{v}_1 + \ldots]$ or equations like $b_1 - 2b_2 + b_3 = 0$ from $[A|\vec{b}]$ reduction.
A basis is $\{$column 1, $\ldots$ of original matrix$\}$ - they span the column space via linear combinations and are l.i.

null space of A=solutions of $A\vec{x} = \vec{0}$. Aug: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Parametrize: $s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. A basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

- $A\vec{x} = \lambda\vec{x}$: matrix multiplication to scalar multiplication
- determinant $(A - \lambda I) = 0$ or Maple for λ
- plug in each λ for nullspace of $(A - \lambda I)$ or use Maple for basis for eigenspace
- $A\vec{x} = \lambda\vec{x}$: A keeps eigenvectors $\vec{x}$ on the same line scaled by λ, so can reason geometrically for transformations

A := Matrix([[21/40,3/20],[19/240,39/40]]);

Eigenvectors(A); $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -6 & \frac{6}{19} \\ 1 & 1 \end{bmatrix}$

The eigenvector decomposition and trajectory:

$$\begin{bmatrix} \text{Owls}_k \\ \text{Rats}_k \end{bmatrix} = c_1 \left(\tfrac{1}{2}\right)^k \begin{bmatrix} -6 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{6}{19} \\ 1 \end{bmatrix}$$