History of the Numerical Methods of Pi

By Andy Yopp
MAT3010 Spring 2003
Appalachian State University

This project is composed of a worksheet, a time line, and a reference sheet. This material is intended for an audience consisting of undergraduate college students with a background and interest in the subject of numerical methods. This material is also largely recreational. Although there probably aren't any practical applications for computing pi in an undergraduate curriculum, that is precisely why this material aims to interest the student to pursue their *own* studies of mathematics and computing.

The time line is a compilation of several other time lines of pi. It is explained at the bottom of the time line and further on the worksheet reference page. The worksheet is a take-home assignment. It requires that the student know how to construct a hexagon within and without a circle, use iterative algorithms, and code in C. Therefore, they will need a variety of materials and a significant amount of time.

The most interesting part of the project was just how well the numerical methods of pi have evolved. Currently, there are iterative algorithms available that are more precise than a graphing calculator with only a single iteration of the algorithm. This is a far cry from the rough estimate of 3 used by the early Babylonians. It is also a different entity altogether than what the circle-squarers have devised. The history of pi and its numerical methods go hand-in-hand through shameful trickery and glorious achievements. It is through the evolution of these methods which I will try to inspire students to begin their own research.

A π Timeline

c. 2000 B.C.E.	Babylonians use $\pi = 3\frac{1}{8} = 3.125$ Egyptions use $\pi = \frac{256}{81} = 3.1605$
c. 1100 B.C.E.	Chinese use $\pi = 3$
c. 550 B.C.E.	*Bible*, I Kings 7:23 implies $\pi = 3$
c. 440 B.C.E.	Hippocrates of Chios squares the lune
c. 434 B.C.E.	Anaxagoras attempts to square the circle
c. 430 B.C.E.	Antiphon and Bryson articulate the principle of exhaustion
c. 420 B.C.E.	Hippias discovers the quadratix
c. 335 B.C.E.	Dinostratos uses the quadratix to "square the circle"
c. 240 B.C.E.	Archimedes uses a 96-sided polygon to establish $\frac{223}{71} < \pi < \frac{22}{7}$ He also uses a spiral to "square the circle."
2nd cent. C.E.	Claudius Ptolemy uses π = 3° 8' 30" = $\frac{377}{120} = 3.1466\ldots$
3rd cent. C.E.	Wang Fau uses $\pi = \frac{142}{45} = 3.155\ldots$ Chung Hing uses $\pi = \sqrt{10} = 3.166\ldots$

263 C.E.	Liu Hui uses $\pi = \frac{157}{50} = 3.14$
c. 450	Tsu Ch'ung-chih finds $\pi = \frac{355}{133}$ by using a circle 10 feet across
c. 530	Aryabhata uses $\pi = \frac{62,832}{20,000} = 3.1416$
c. 650	Brahmagupta uses $\pi = \sqrt{10} = 3.166...$
1220	Leonardo de Pisa (Fibonacci) finds $\pi = 3.141818...$
1429	Al-Kashi calculates π to 16 decimal places
1573	Valentinus Otho finds $\pi = \frac{355}{113} = 3.1415929$
1593	Francois Viete expresses pi as an infinite product using only 2s and πs ; Adrien Romanus finds π to 15 places
1596	Ludolph van Ceulen calculates π to 35 places
1621	Willebrod Snell refines Archimedes method
1630	Grienberger uses Snell's refinement to calculate π to 39 decimal places
1654	Huygens proves the validity of Snell's method
1655	John Wallis finds an infinite rational product for π ; Brounker converts it to a continued fraction

1663	Murmatsu Shigekiyo finds seven accurate digits in Japan
1665-66	Isaac Newton discovers calculus and calculates π to at least 16 decimal places; This was not published until 1737
1671	James Gregory discovers the arctangent series
1674	Gottfried Wilhelm Leibniz discovers the arctangent series for pi; $\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
1699	Abraham Sharp uses Gregory's series with $x = \sqrt{3}$ to calculate π to 71 places
1706	John Machin calculates π to 100 places; William Jones uses the symbol " π " to describe the ratio between the circumference of a circle and its diameter
1713	Chinese court publishes *Si-li Ching-yun* which shows π to 19 digits
1717	Abraham Sharp finds π to 72 places
1719	Thomas Fantet de Lagny calculates π to 127 places
1722	Takebe Kenko finds 40 digits in Japan
1736	Leonhard Euler proves that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \ldots = \frac{\pi^2}{6}$

1737	Euler uses the symbol π, thus establishing it as standard notation
1748	Euler publishes the *Introductio in analysin infinitorum* containing Euler's theorem and many series for π
1755	Euler derives a very rapidly converging arctangent series
1761	Johann Heinrich Lamber proves that π is irrational
1767	Euler suggests that π is transcendental
1777	Comte de Buffon devises his needle problem, introducing probability theory to the world of π
1794	Georg Vega calculates π to 140 decimal places; A. M. Legendre proves the irrationality of π and π^2
1840	Liouville proves the existence of transcendental numbers
1844	L. K. Shulz von Stssnitzky and Johann Dase calculate π to 200 places in under 2 months
1855	Richter calculates π to 500 decimal places
1873	Charles Hermite proves the transcendence of π
1873-4	William Shanks calculates π to 707 decimal places
1874	Tseng Chi-hung finds 100 digits in China
1882	Ferdinand von Lindermann shows that π is transcendental, so the circle cannot be squared

1945	D. F. Ferguson finds Shanks's calculation wrong at the 527th place onwards
1947	Ferguson calculates 808 places using a desk calculator, a feat that took about a year
1949	ENIAC computes 2,037 decimal places in seventy hours
1955	NORC computes 3,089 digits in thirteen minutes
1957	Pegasus computer (London) computes 7,480 places
1959	IBM 704 (Paris) computes 16,167 places
1961	Daniel Shanks and John Wrench use IBM 7090 (New York) to computer 100,200 decimal places in 8.72 hours
1966	IBM 7030 (Paris) computes 250,000 decimal places
1967	CDC 6600 (Paris) computes 500,000 decimal places
1973	Jean Guiloud and M. Bouyer in use a CDC 7600 (Paris) to compute 1 million decimal places in 23.3 hours
1976	Salamin and Brent find an arithmetic mean algorithm for π
1983	Y. Tamura and Y. Kanada compute 16 million digits in under 30 hours
1988	Kanada computes 201,326,000 digits on a Hitachi S-820 in six hours
1989	Chudnovsky brothers find 480 million digits; Kanada calculates 536 million digits; Chudnovsky brothers calculate 1 billion digits

1995	Kanada computes 6 billion digits
1996	Chudnovsky brothers compute over 8 billion digits
1997	Kanada and Takahashi calculate 51.5 billion digits in just over 29 hours
1999	Kanada calculates over 206 billion digits using the Gauss-Legendre algorithm
What's Next???	

This time line was compiled from three different sources that were all very similar, yet differed in small aspects. They were: The Joy of Pi, A History of Pi, and A Piece of Pi. Please refer to the worksheet resource page for more information on these books.

Introduction to the Numerical Methods of pi Worksheet

This worksheet will introduce you to some of the ways in which people calculate π . Currently, people can calculate π correctly to more than 50 billion digits using computers. Why do they do it? Well, its hard to explain, but understanding and calculating this mysterious number has captivated people for more than 2000 years. With this much time devoted to the problem, many methods have emerged. Very often those methods have matured and lent insights to better ones as time went on. Therefore, to better understand the methods that we use today, lets look at how it was done in the past. Have the π time line, some paper, a calculator, and a compass handy. Later, be ready to write a small program.

1. Briefly study the π time line.

2. History's first scientific attempt to compute π was done by Archimedes of Syracuse in about 240 B.C.E. Recall the upper and lower bounds which he found for π . On a blank piece of paper, draw a circle and construct a hexagon inside of it. Next, draw the radii of the circle to two adjacent vertices of the hexagon, forming an equilateral triangle with one side of the hexagon. Find the perimeter of the hexagon ___________ and the circumference of the circle __________. Write an inequality that describes the numbers you just found ________ >________ and solve for π : π > _________. Next, draw a circumscribed hexagon about the circle. In the new hexagon, construct an equilateral triangle as before. If *r* is the side of the triangle formed by the radius, and *h* is the shortest side, $r = \sqrt{(3 \times h)}$. Solve for h ________. Now find the perimeter of the new hexagon ________. Substitute the value of h to get a formula for the perimeter in terms of *r* ________. Write an inequality expressing the relationship between the circumference of the circle and the perimeter of the circumscribed hexagon ________ < ________ and solve for π : π < ________. Now combine both of your inequalities ________ < π < ________

   Compare these to Archimedes' upper and lower bounds. He used 96-gons! Could you code this?

3. Aryabhata was a prominent Indian mathematician. He used the following method for finding π between 500 and 1000 C.E.:

   "Add 4 to 100, multiply by 8, and add again 62,000. The result is approximate value of the circumference when the diameter is twenty thousand."

   Decipher the statement and write it as a formula ____________.

   Use the formula you just wrote to estimate π . What do you get? ________.

4. In 1655, John Wallis discovered an infinite product that equaled π :

$$\frac{\pi}{2}=\frac{(2\times2\times4\times4\times6\times6\times8\ldots)}{(1\times3\times3\times5\times5\times7\times7\ldots)}$$

This was revolutionary because the irrational number π could be expressed as products of the even and odd integers! Try calculating a few terms of this product ________. Why or why wouldn't you want to use this on a computer? ____
____.

5. Recall from the time line that James Gregory and Gottfried Leibniz independently discovered the following infinite series:

$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}\ldots$$

The trigonometric argument behind this formula relies on the fact that the tangent of 45 degrees is one. 45 degrees equals $\frac{\pi}{4}$ in radians, therefore, if you take the arctangent of one, you get $\frac{\pi}{4}$ radians. Using a calculator, try arctan(1) using the above formula ________. Add 6 more terms onto the formula to get another approximation ________. How many accurate digits do you have? ________. Why or why wouldn't you use this algorithm to approximate π using a computer? Can you guess how fast it would converge?
____
____.

From the beginning of the 18th century, π was calculated using John Machin's formula: $\frac{\pi}{4}=\arctan(\frac{1}{5})-\arctan(\frac{1}{239})$. This formula was a lot more powerful than the Gregory-Leibniz series because it converged much faster. Extensions to Machin's formula have been used on supercomputers to calculate π to more than one million places. Since then, even more powerful formulas have been devised. Now we will introduce some modern methods.

6. In 1914, Indian mathematician Srinivasa Ramanujan found this formula:

$$\frac{1}{\pi}=\frac{(2\sqrt{2})}{9801}\sum_{n=0}^{\infty}\left[\frac{\frac{4n!}{(n!)^{4}}\times[1103+26390n]}{(4\times 99)^{4n}}\right]$$

Try plugging n = 0 into your calculator and write down the result _____________.

Keep in mind that we haven't even reached the computer age yet!

7. In the 1990's the Chudnovsky Brothers developed this formula:

$$\frac{1}{\pi}=12\times\sum_{n=0}^{\infty}(-1)^{n}\times\left(\frac{6n!}{((n!)^{3}3n!)}\right)\times\left(\frac{(13591409+545140134n)}{640320^{(3n+\frac{3}{2})}}\right)$$

Again, plug n = 0 into your calculator and write down the result_____________. Are you surprised?

8. On September 18, 1999, Yasumasa Kanada used the Gauss-Legendre method which uses the arithmetic geometric mean to calculate π to more than *eight billion* digits on one of the world's most powerful computers! Here is the algorithm:

$$a_{0}=1\,;b_{0}=\frac{\sqrt{2}}{2}\,;t_{0}=\frac{1}{4}\,;x_{0}=1$$

$$a_{(n+1)}=\frac{(a_{n}+b_{n})}{2}\,;b_{(n+1)}=\sqrt{(a_{n}b_{n})}$$

$$t_{(n+1)}=t_{n}-x_{n}(a_{(n+1)}-a_{n})^{2}\,;x_{(n+1)}=2x_{n}$$

$$\text{then}\quad \lim_{(n\to\infty)}\left(\frac{(a_{n}+b_{n})^{2}}{4t_{n}}\right)=\pi$$

Write a C program that performs this algorithm. Run it, and comment on how fast it converges ____
____.

How many iterations did it take to converge?_____________________________.

9. On your own, find 3 formulas or algorithms that have not been listed on this worksheet. Code them in C, and comment on their performance.

So how do you like π ? Maybe by now you can get a feel for why so many people are endlessly calculating this mysterious number. Actually, the reason that most people study π is because they are looking for patterns and order in the seemingly random digits. π is a fundamental building block of the universe. You could say that those who study π are trying to "read between the lines" of the very fabric of the universe, and as we can see from the time line, this search has been going on for a very long time. Although no relevant patterns have been found in π and it has not yet been proven that its digits are random, here are some interesting facts that might help to accelerate your own studies of π !

Hint for number 5: In order to calculate 100 digits of π using the Gregory-Leibniz series, you would have to calculate more terms than there are particles in the universe!

Since there are 360 degrees in a circle, look at the 360^{th} digit of π – the number 360 is centered over that digit!

The sequence 3333333 appears at the $710{,}100^{th}$ digit and again at the $3{,}204{,}765^{th}$ digit.

The sequence 0123456789 occurs in the decimal expansion of π six times in the first 50 billion digits; the sequence 9876543210 occurs five times, and the sequence 27182818284 (*e*) occurs at the $45{,}111{,}908{,}393^{rd}$ decimal place!

Resource List

Blatner, David. The Joy of Pi. Walker Publishing Company, Inc.

The Joy of Pi is an amazing book that is aimed at more of a leisurely audience. It contains facts about pi and its history, its calculation, trivia, and those who have participated in its history.

Harris, Herman H. Jr. "The History and Calculation of Pi." The Emporia State Research Studies 8 1 (1959).

This article originated as a Master's Thesis by Herman Harris Jr. at Kansas State Teacher's College. It is about 35 pages long.

Beckmann, Petr. A History of Pi. Boulder, Co: The Golem Press, 1971.

This is one of the most thorough and in-depth books out there on pi. It is one of the most frequently referenced books that you will see in other studies on the history of pi. This book delves into the rich history of the number and those who worked on it tirelessly. Additionally, it contains much of the math that goes along with that history. Note, there are updated editions of this book.

Bokhari, Nailia. Piece of Pi. Dandy Lion Publications, 2001.

This is a wonderful activity book for teachers. It is aimed at grades 6-8, but I found a lot of useful material in it. It is seems to be a conglomeration of material from The Joy of Pi and A History of Pi, so in some ways it was difficult to differentiate information, and occasionally there was information like dates that did not agree. But still, its a unique and useful book.