Worksheet and references

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The fifth postulate and Non-Euclidean Geometry

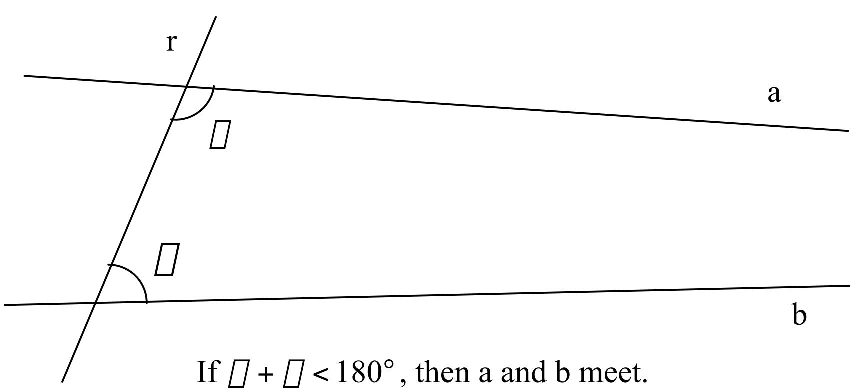
At the beginning, geometry was an experimental science, that is, all its results were achieved from experiments, or from applications of geometry in the real life of ancient people. But, the study of geometry went in another direction, when one noticed that some propositons did not need experimental arguments, becuase they could be proved from others propositions using logical thought. I am going to present the fifth postulate, and a kind of non-Euclidean geometry where this postulate is false, and I am going to point out some differences between this geometry and Euclidean geometry.

The first systematic and constructive exposition of geometry can be found in “THE ELEMENTS” of Euclid (300 B.C.). In these thirteen books, Euclid considers five postulates that cannot be proved, and that are assumed as absolute truth, and his geometry is based on these five postulates.

Euclid tried to choose as postulates those affirmations that for their simplicity could be accepted by anyone with good judgment, and that in some sense, they were evident by themselves.

The first four postulates satisfy effectively the above mentioned conditions of simplicity and evidence, however the fifth, the last one and the most controversial among them is presented by Euclid as follow:

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



If $\alpha + \beta < 180^\circ$, then a and b meet.

We think that even Euclid noticed that the fifth postulate was not self-evident. This affirmation is confirmed because he avoided its use as long as possible. For example, it is possible to prove the three classical cases of congruent triangles, without make use of the fifth postulate.

There exist a lot of equivalent ways to present the fifth postulate, but among them the most famous is that given by Playfair:

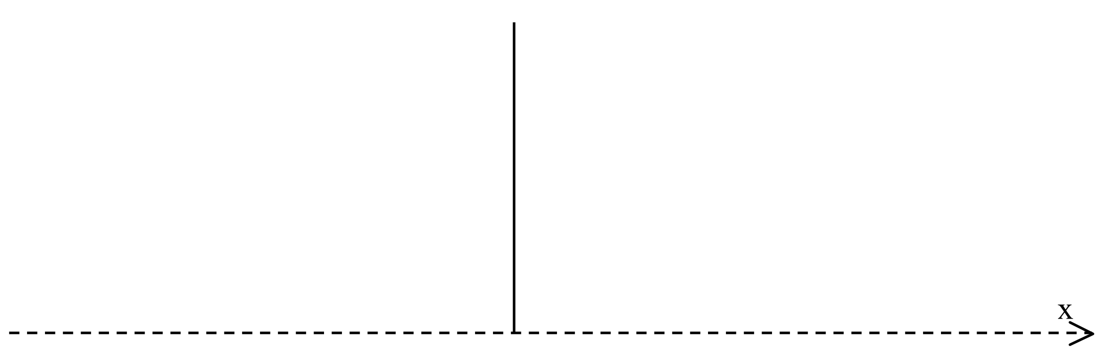
"Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line".

(Euclid defines parallel lines as that ones that produced indefinitely do not meet).

As we can see, the fifth postulate, or the parallel postulate, is not self-evident. For more than 2000 years, many mathematicians tried to prove the fifth postulate from the other four, but all attempts assumed an equivalent argument, or gave a false proof. This led to the development of Non-Euclidean geometries. The existence of such geometries shows that the fifth postulate does not depend of the other four. One example of Non-Euclidean geometry is Hyperbolic geometry, and I am going to show Poincarè's model for it. In this geometry all Euclid's postulates are verified, except the parallel one.

Poincarè's model for Hyperbolic Geometry

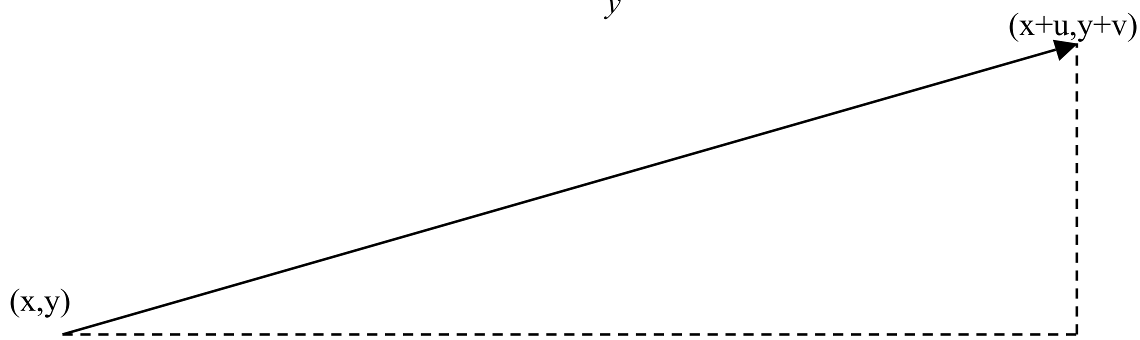
Our space is the opened upper half-plane: $(R^2)^+ = \{(x, y) \in R^2 \mid y > 0\}$.



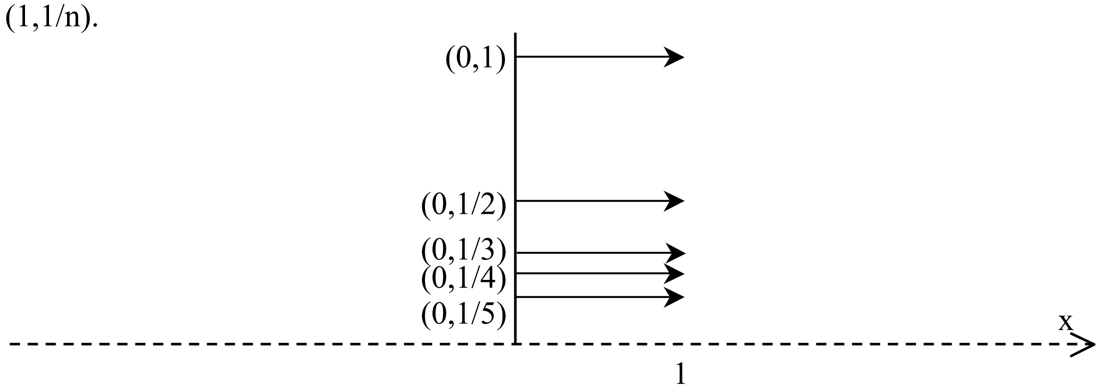
How to measure vectors in Hyperbolic Geometry

In $(R^2)^+$ we introduce the following method to measure the length of a vector: If the vector w begins at (x,y) and has components (u,v), then the length $\|w\|$ of w is defined by

$$\|w\| = \frac{\sqrt{u^2 + v^2}}{y}.$$



We can notice the difference between the Euclidean metric $(\|w\| = \sqrt{u^2 + v^2})$ and the hyperbolic metric is the denominator y that appears in the above definition. Intuitively, it is as if we measured length of vectors with a changeable ruler, which is exactly the Euclidean ruler over the line y=1. To illustrate the difference between the hyperbolic metric, and the Euclidean metric, we can consider the vectors w_n from (0,1/n) to (1,1/n).



We have that in the Euclidean metric all these vector have length 1, but in the hyperbolic metric,

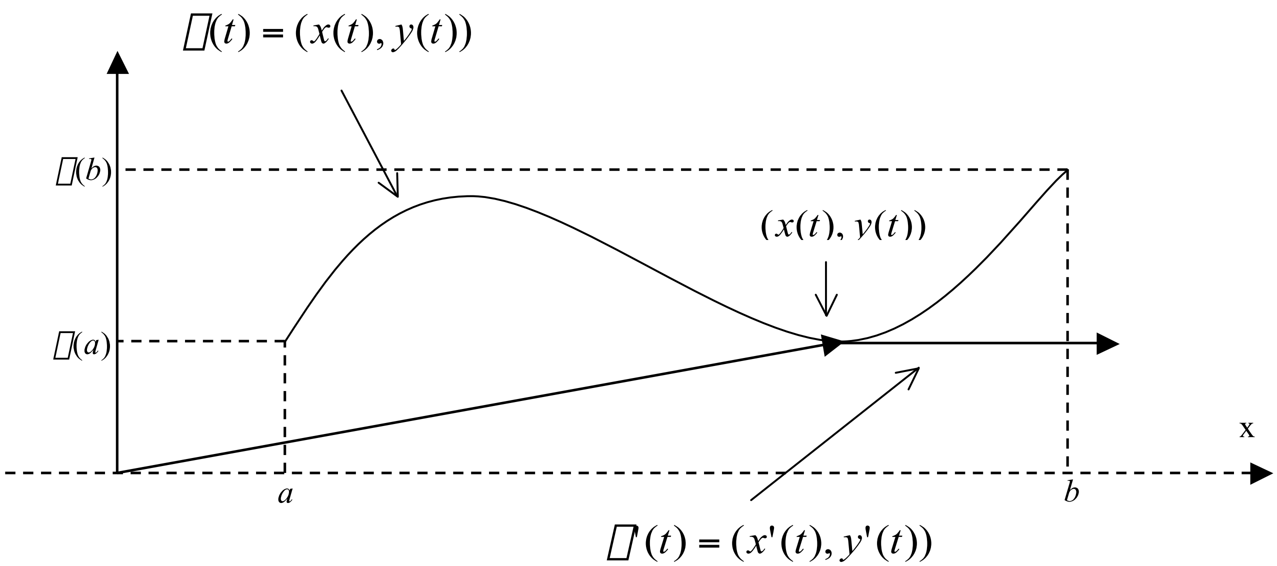
$$\|w_n\| = \frac{\sqrt{1^2 + 0^2}}{1/n} = n,$$

that is, the length of the vectors changes, and it goes to infinity as this vectors approach x-axis.

I am going to indicate $(R^2)^+$ with above defined metric by H.

Method to measure length of Curves

The length of a curve is defined as usually by the integral of the length of the tangent vector.



That is, if ℓ is the length of the curve from $(a,\gamma(a))$ to $(b,\gamma(b))$, then

$$\ell = \int_a^b \|\gamma'(t)\| dt = \int_a^b \|(x'(t), y'(t))\| dt = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)} dt$$

Distance between points

The distance between two points $P, Q \in H$ is defined by the infimum of the length of the curves that connects P to Q, and a curve whose length is the infimum is called a geodesic of H.

It is possible to prove that the geodesics of H are Euclidean semicircles with center on the x-axis and, straight lines orthogonal to the x-axis, on the other hand, the geodesics in the Euclidean geometry are straight lines.

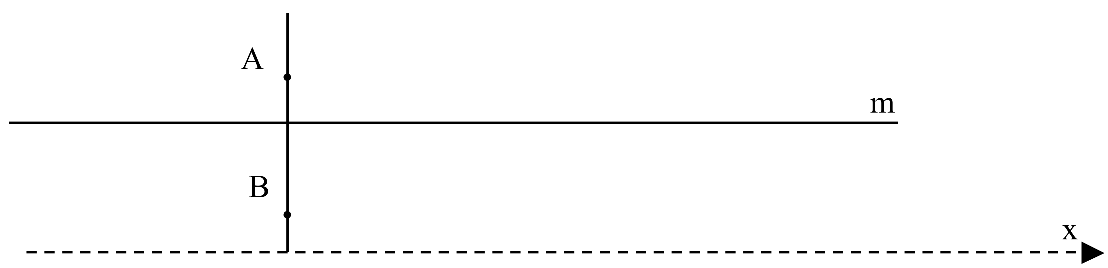
A good question is:

How does one construct a geodesic given two points $A, B \in H$?

The answer is: Construct m, the Euclidean straight line orthogonal to the Euclidean segment AB, through its middle point.

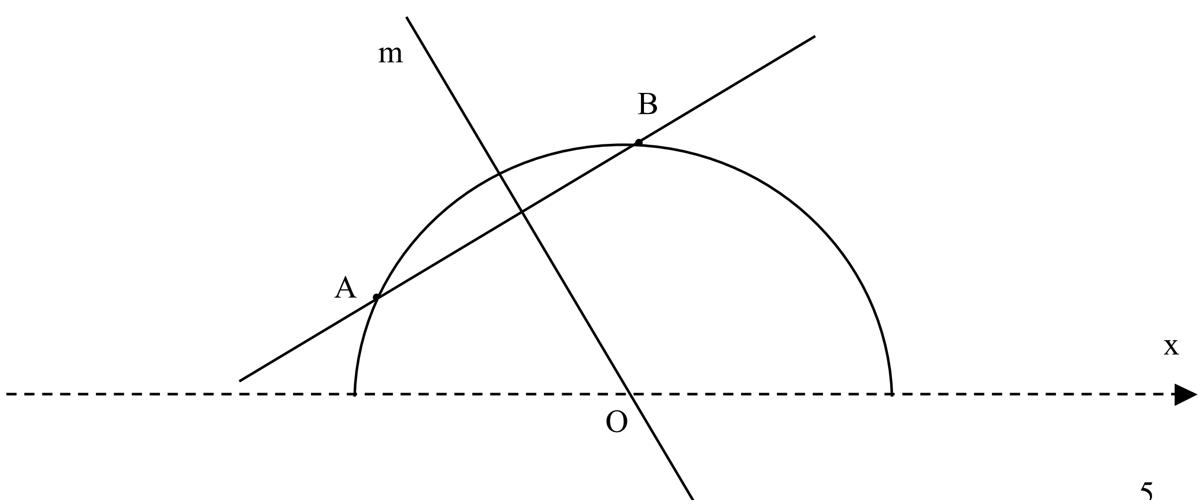
First case: m is parallel to the x-axis

Then, from Euclidean geometry we know that there exists a unique Euclidean straight line through A and B, which is orthogonal to the x-axis.



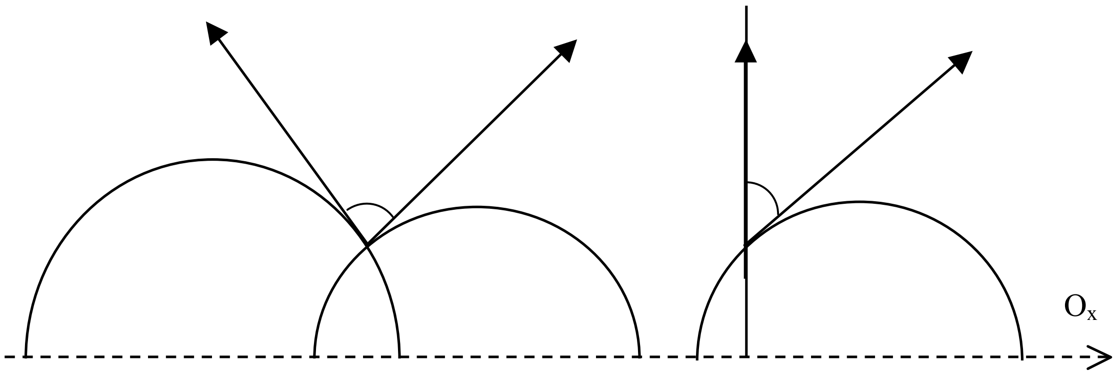
Second case: m meets the x-axis at O

Again, by means of Euclidean geometry, we know that there exists a unique Euclidean semicircle through A and B, with center O (Since $O \in m$, and m is the set of the points that are at the same distance of A and B.).



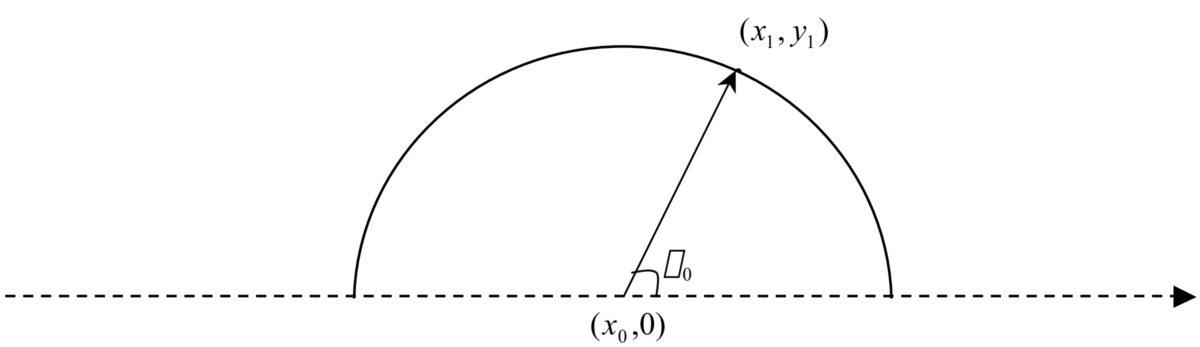
Angle between lines

The angle between two hyperbolic lines is defined by the Euclidean angle between the respective tangent vectors.



As we know, in Euclidean geometry, if it is given a positive real number m and a point P on a straight-line r, it is possible to construct, from this point, two segments of length m (one to the right, and other one to the left). To show that it is also true in hyperbolic geometry, it is enough to verify that the two types of lines (Euclidean straight lines orthogonal to x-axis and Euclidean semicircles with center on x-axis) have infinite length.

First Case: The line is a semi-circle with center $(x_o,0) \in x-axis$ and (Euclidean) radius k .



Writing

$$\begin{cases} x = x_o + \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

we get polar coordinates (ρ,θ) on H. It follows that $dx = -\rho\sin\theta\, d\theta$. So, if ℓ is the hyperbolic length of the segment from $P_1=(x_1,y_1)$ on the line (on the semicircle) to the limit (to the left side) of this line with the x-axis, we have

$$\ell = \left| \int_{x_1}^{x_o-k} \frac{\sqrt{1+\left(\frac{-(x-x_o)}{\sqrt{k^2-(x-x_o)^2}}\right)^2}}{\sqrt{k^2-(x-x_o)^2}}\,dx \right| = \left| \lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon} \frac{\sqrt{1+\frac{(\rho\cos\theta)^2}{(\rho\sin\theta)^2}}}{(\rho\sin\theta)}(-\rho\sin\theta)\,d\theta \right| =$$

$$= \left|\lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon}\sqrt{1+\cot^2\theta}\,d\theta\right| = \left|\lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon}\csc\theta\, d\theta\right| = \left|\lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon}\frac{d\theta}{\sin\theta}\right|.$$

Where I used that $|\csc\theta| = \csc\theta$, since $0<\theta<\pi$. On the other hand, we know that $\csc\theta = \frac{1}{\sin\theta} \geq \frac{\cos\theta}{\sin\theta}$, so we have

$$\ell = \left|\lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon}\frac{d\theta}{\sin\theta}\right| \geq \left|\lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon}\frac{\cos\theta}{\sin\theta}\,d\theta\right| = \left|\lim_{\varepsilon\to 0}\int_{\theta_o}^{\pi-\varepsilon}\cot\theta\, d\theta\right| = \left|\lim_{\varepsilon\to 0}\ \ln|\sin\theta|\Big]_{\theta_o}^{\pi-\varepsilon}\right| =$$

$$\left|\lim_{\varepsilon\to 0}\ln|\sin(\pi-\varepsilon)| - \ln|\sin\theta_o|\right| = \left|-\infty - \ln|\sin\theta_o|\right| =$$

Second case: The line is a Euclidean straight line orthogonal to the x-axis

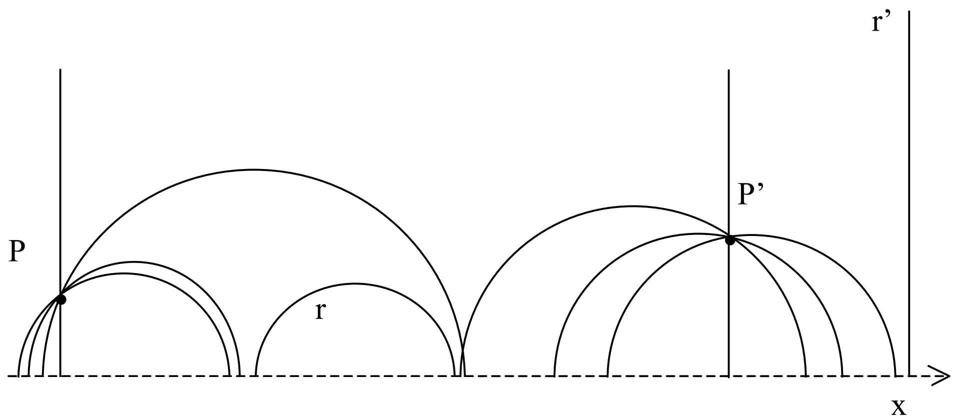
In this case, we have that $x = x_o$. We can see that "up" the line goes to infinity, so we just need to show that the same occurs "down", that is, when this line goes to O_x its length goes to infinity. So, let $P_o = (x_o, y_o)$ be a point on this line, the length ℓ from P_o to the limit x-axis is given by

$$\ell = \left|\lim_{\theta \to 0} \int_{y_o}^{y} \frac{1}{y}\,dy\right| = \left|\lim_{\theta \to 0} \ln\left|\sin\theta\right|\Big|_{\theta_o}^{\theta}\right| = \left|\lim_{\theta \to 0} \ln\left|\sin\theta\right| - \ln\left|\sin\theta_o\right|\right| = \left|-\infty - \ln\left|\sin\theta_o\right|\right| =$$

Therefore, it is proved that lines in hyperbolic geometry have infinite length. And now, I am going to present the hyperbolic version of the fifth postulate.

Hyperbolic version of the fifth postulate

"Given a line and a point not on the line, it is possible to draw an infinite number of lines through the given point parallel to the line.**"**



In virtue of this version to the parallel postulate, in hyperbolic geometry some properties are obtained with a different report with relation to the same properties in Euclidean geometry. For example, in Euclidean geometry we have the following proposition:

"The sum of the internal angles of a (Euclidean) triangle is 180°".

This same proposition in hyperbolic geometry is presented as follows:

"The sum of the internal angles of (hyperbolic) triangle is less than 180°".

Other differences between Euclidean geometry and hyperbolic geometry occur with relation to similar triangles. In Euclidean geometry there exist three different conditions for two triangles to be similar, and there exist triangles that are similar and not congruent (equal). But, in hyperbolic geometry, there are no triangles which are only similar, in the sense that, in hyperbolic geometry similar triangles are also congruent.

REFERENCES

[1] – Carmo, M. P. – Geometria Não-Euclidiana, Matemática Universitária, N-6, 1987.

I found some comments about Poincarè's model for Hyperbolic Geometry.

[2] – H. S. M. Coxeter, F. R. S., Non-Euclidean geometry. University of Toronto Press, Great Britain 1961. I found some observations with relation to the differences between Euclidean geometry and Hyperbolic Geometry.

[3] – Wolfe, Harold E., Introduction to non-Euclidean geometry. The Dryden Press, New York, 1948. I used it on the foundations of Euclidean geometry; the fifth postulate; the discovery of non-Euclidean geometry; and hyperbolic plane geometry.

[4] – Greenberg, M. J., Euclidean and non-Euclidean geometry, development and history. W. H. Freeman and Company, New York, 1980. I found on it some results about hyperbolic geometry.

[5] – www.members.tripod.com/~noneuclidean/history.html

It contains a program where we can draw some picture in a circle, what is the Poincaré's model (in a circle) for Hyperbolic Geometry.

[6] – http://cs.unm.edu/~joel/NonEuclid/

It contains both disk and upper half-plane plane models to Hyperbolic Geometry.

[7] –http://cvu.strath.ac.uk/courseware/msc/jgraves/HyperbolicGeometry.html

It contains a list of theorems in Hyperbolic Geometry.