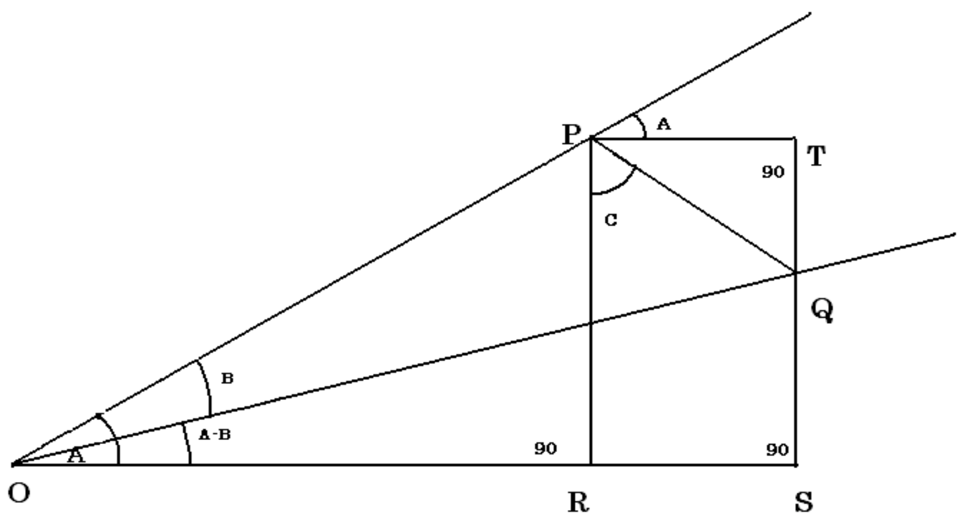
Ptolemy's Theorem

Ptolemy was an astronomer and mathematician in the second century. He studied the stars and the position of the planets. His math shows us that he thought the earth was stationary. Even though he wasn't right, he developed math we still use today. By using chords of circles, he created a table of values for what we now call sine and cosine. Ptolemy did not call them sine and cosine. Hindu mathematicians used *jya* for sine, which was translated into *jiba* in Arabic, which has no meaning. Later, through many translations, the word became *jaib*, which means fold. When translated into Latin, this word became sinus. Fibonnaci was the first to begin calling this trigonometric function sine. We'll call them sine and cosine for our own benefit. He made a table of sines from 0 to 90 degrees, going up a half degree each time. In addition, he came up with a theorem to find the sine of two angles added or subtracted. Later on, you will derive part of Ptolemy's theorem by yourself. Remember sine = $\frac{opposite}{hypotenuse}$ and cosine = $\frac{adjacent}{hypotenuse}$.

PART I

Here is an example of the method Ptolemy used to come up with the formula sin (A - B) = (Sin A * Cos B) – (Sin B * Cos A).



(Figure not to scale)

1. Sin A = $\frac{PR}{OP}$ Sin B = $\frac{PQ}{OQ}$ (Since $\angle QPO$ is 90°) Cos B = $\frac{OP}{OQ}$

   Since (90 – A) + C = 90, A = C, $\therefore$ Cos C = Cos A = $\frac{TQ}{PQ}$

2. Notice that PR $\cong$ TS since figure TPRS is a rectangle, so

$$\text{Sin (A – B)} = \frac{QS}{OQ} = \frac{TS - TQ}{OQ} = \frac{PR - TQ}{OQ} = \frac{PR}{OQ} - \frac{TQ}{OQ}$$

3. Now take a minute and just look at $\frac{PR}{OQ}$. Multiplying by $\frac{OP}{OP}$ is just like multiplying by 1. Also note that multiplication is commutative, so you can change the order, like I did in the denominator.

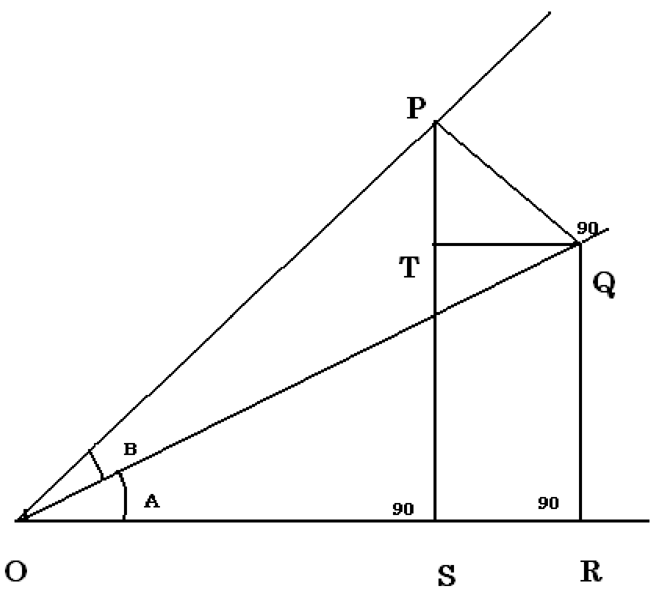
$$\frac{PR}{OQ} * \frac{OP}{OP} = \frac{PR * OP}{OQ * OP} = \frac{PR * OP}{OP * OQ} = \frac{PR}{OP} * \frac{OP}{OQ}$$

4. With that same logic, $\frac{TQ}{OQ} = \frac{PQ}{OQ} - \frac{TQ}{PQ}$.

So, $\text{Sin (A – B)} = \frac{PR}{OP} * \frac{OP}{OQ} - \frac{PQ}{OQ} * \frac{TQ}{PQ}$.

5. Hence, Sin (A – B) =(Sin A * Cos B) – (Sin B * Cos A).

PART II
Whew, ok, so now it’s your turn.



(Figure not to scale)

It’s not clear from the figure, but $\angle PQO = 90°$. You’ll use this soon.
Fill in the blanks as you go.

1. Sin A = $\frac{\quad}{OQ}$ Cos A = $\frac{OR}{\quad} = \frac{PT}{PQ}$ (Since $\angle OQT = \angle A$)

   Sin B = $\frac{\quad}{OP}$ Cos B = $\frac{OQ}{\quad}$

2. Sin (A + B) = $\frac{PS}{OP}$, PS is made of two segments, substitute those in.

   Sin (A + B) = .

3. Notice that figure RQTS is a rectangle, so QR $\cong$ TS. Substitute this in.

   So, Sin (A + B) = $\frac{QR + PT}{OP}$. Split the fraction into two fractions.

4. Sin (A + B) =

Remember step 2 from the last example? The same idea follows here. These next two steps do not follow each other, they just have the same idea. Step 7 will follow steps 5 and 6.

5. $\frac{QR}{OP} * \frac{OQ}{OQ} = \frac{QR * OQ}{OP * OQ}$.

Use the commutative property of multiplication (a*b=b*a) to reverse the order in the denominator.

$= \frac{QR * OQ}{\quad}$

6. $\frac{PT}{OP} * \frac{PQ}{PQ} = \frac{PT * PQ}{OP * PQ}$

Use the commutative property of multiplication to reverse the order in the numerator.

7. Now compare your results from #5 and #6 to your results from #1. Your answer will contain sine and cosine functions.

Sin (A + B) =

8. Self Check: look back at the figure. What is the Sin (A + B)? Does your answer you wrote for #7 equal the answer you get from the picture? Show me.

Congratulations! You just derived Ptolemy's Theorem, a theorem over 1800 years old, by yourself, and you're only in high school!

References:

This worksheet was made primarily from *Mathematics in the Making* written by Lancelot Hogben, and published in 1960 by Doubleday. This book supplied the figures and a bare bones proof of Ptolemy's theorem.

The origin of the word sine comes from http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html.

Also, this worksheet was made with assistance from Dr. Sarah, Dr. Rhodes, Ederson Moreira Dos Santos, and Ben Wooten.