Foundations of Probability Worksheet

Pascal

The basis of probability theory can be traced back to a small set of major events that set the stage for the development of the field as a branch of mathematics. The origins of probability as it is defined today as the measure of likelihood can be attributed almost completely to the genius of Blaise Pascal. His solution to the problem of points, which we will discuss, was perhaps the single most significant and inspiring event to occur in the history of probability. His applications of probability to dice games and other games virtually defined the bridge between mathematics and chance. In addition, his work with the binomial theorem and binomial coefficients and their application to the field of probability outlined an immeasurably important element of the subject. These concepts are presented here with brief passages of their historical foundation as well as some active exercises relative to each topic.

I. Problem of Points

Archaeological evidence has been found to suggest that games of chance or gambling have been popular among many cultures since the beginning of civilization. It was not until the 17th century, though, that any mathematical developments in the analysis of chance occurred. The birth of probability is often attributed to Blaise Pascal's solution to the problem of points in 1654. The problem is best represented by a game in which two players gamble on who obtains first the set amount of points, which are scored by a fair coin landing on a player's designated side. The players flip the coin until a certain number of heads or tails occur and each given a point for their chosen outcome. So if the desired amount of points is ten, and one player has 7 points and heads and the other has 8 points and tails, how would the money be divided if the game was interrupted at this point? This solution to this question is very likely the foundation of probability theory.

1. Consider the coin toss game in which the object is ten points. Say player 1 has heads and 7 points and player two has tails and 8 points. The game would be over in at most four tosses of the coin because with only two outcomes, either three heads or two tails will certainly occur in four tosses. Write out all the possibilities of outcomes for four tosses, denoting heads by 'h' and tails by 't'.

2. How many of the outcomes for four tosses resulted in the player with heads winning the bet? How many resulted in tails winning?

3. Divide the number of winning outcomes for heads by the total number of outcomes and then do the same for tails.

4. Multiply each result in (2) by the amount of money at stake (assume $100).

These are the correct amounts of money each player should receive. With this solution, Pascal initiated the exploration into the field of probability theory.

II. Dice Games

In the same year as Pascal's solution to the problem of points, another game of chance sparked a correspondence between Pascal and another notable mathematician, Pierre Fermat. The problem arose from an inquiry to Pascal from a French nobleman, the Chevalier de Méré. The Chevalier was of the most famous gamblers of the 17^{th} century . He was partial to dice games and frequently would offer the bet that he could obtain a 6 in four rolls or less. The Chevalier would win this game more often than not, which he mistakenly attributed to having a 4/6 or 2/3 chance of winning.

1. Play the Chevalier's game by rolling a die four times, record whether you win or lose, and repeat the process for five games.

Game: 1. **W L**
2. **W L**
3. **W L**
4. **W L**
5. **W L**

2. Would you wager money on this game, if you were the roller? Or the contestant?

After his bettors became aware that they were more likely to lose, they became fewer and the Chevalier was forced to devise a new dice game that would attract more gamblers. Now he proposed that he would offer an even money bet that he could roll a twelve with two dice in 24 rolls or less. Using the same misled reasoning as he used to calculate the odds of winning the first game, he figured that the chance was 24/36 or 2/3 of winning.

3. Roll a pair of dice up to 24 times to see if a 12 is rolled.

In this case, though, he began losing more often than not and in his frustration, consulted Pascal. This problem led to Pascal's correspondence with Fermat. They began to analyze the problem mathematically. They considered probabilities of both problems.

1. What is the chance of getting a 6 in one roll of a fair die?

2. What is the chance of not rolling a 6? (1- answer in (1))

3. Take the answer in (2) and raise it to the fourth power to correspond to four rolls.

4. Subtract this number from 1 to get the probability of rolling at least one 6 in four rolls.

5. Use the same method to calculate the chance of getting a 12 in twenty four rolls of two dice.

This is the method that Pascal and Fermat used to solve the Chevalier's problem. Their association resulted in five letters published in 1654. The letters are the first publication to explore the rigorous mathematics of probability and are said to lay the foundation for probability theory. It is interesting to note that if the Chevalier had proposed a game in which 25 rolls were made instead of 24, he would have profited and never consulted Pascal. If this had occurred, the history and development of probability theory would be significantly different.

6. What would the chance of the Chevalier winning the same bet, but with 25 rolls instead of 24?

III. The Binomial Theorem and Pascal's Triangle

The binomial theorem has an important link to probability theory. It is an essential element in the calculation of probabilities. It is an algebraic formula for binomial expansions. The theorem states:

For any numbers a and b and any positive integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

The formula for finding all binomial expansions for all values of n was found by Newton, but Blaise Pascal is accredited with yet another landmark in the history of mathematics: the application of this theorem to the world of probability. In 1654, the same year as his vastly important works on games of chance, he also found the time to write *Traité du triangle arithmétique.* This treatise includes relations between and applications of binomial coefficients. Though the expansion of binomial coefficients and perhaps the use of a triangle to represent them date as far back as 11^{th} century Persia and China, Pascal's extensive systematic analysis of the subject was enough to connect his name in the term "Pascal's Triangle". The triangle is a numerical table of the binomial coefficients in the expansion of $(a + b)^n$. It looks like:

n =0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
	. . . . .
	. . . . .

In a section of the Pascal's treatise, he applies the binomial coefficient triangle and the binomial theorem to games of chance and thus probability:

If a and b are complimentary probabilities (their sum is 1), and a is the probability of success, then $(a+b)^n = 1$. Each term of the sum can be interpreted as respectively the probability of k = (0, 1, 2, … n) successes in n independent trials. The triangle is then used to easily obtain the coefficients for each term.

ex. If a fair coin is flipped 4 times, calculate the probability that 3 heads will show up. The probability of the coin landing on heads, a, is equal to that of landing on tails, b, and both are equal to .5. The probability that k = 3 for n = 4 tosses is easily calculated using the binomial theorem and Pascal's triangle. Using these tools, a table can be constructed for n = 4 as follows:

k =

0	1	*	a^0	*	b^4
1	4	*	a^1	*	b^3
2	6	*	a^2	*	b^2
3	4	*	a^3	*	b^1
4	1	*	a^4	*	b^0

The first column represents different values for k. The second column represents the coefficients for each term acquired from Pascal's triangle. The powers of a and b are given by a^k and b^{n-k}. Since we want to know what the probability of getting three heads, we look at k = 3 on the table and see the term

$$4 * a^3 * b^1.$$

Since a and b are both .5, we obtain

$$4 * (.5)^4,$$

which is equal to .25 or _.

1. What is the probability of getting 3 heads if a fair coin is flipped 5 times? (hint: a and b are both .5, n = 5, k = 3)

Besides games of chance, Pascal's triangle and the binomial theorem have many practical applications in the real world:

ex. If the probability of getting audited by the IRS is .06, then what is the probability that out of four randomly chosen people one will be audited? Using the same table as above for n = 4 and k = 1, we get:

$$4 * a^1 * b^3.$$

Since .06 is the probability of "success" or getting audited, .94 is the probability of "failure" or not getting audited. So .06 is assigned to a and .94 to b, and the result is:

$$4 * (.06)^1 * (.94)^3,$$

which is equal to .199.

2. If the probability of having a certain disease is .17, what is the probability that out of 4 randomly chosen people two will have the disease?
(hint: assign .17 to a and 1 - .17 to b, n = 4, k = 2)

Worksheet Goals:

The purpose of this worksheet is to familiarize the student with the foundations of probability. By providing brief historical passages followed with reasonably uncomplicated exercises, the student is able to get an adequate overview of the subject as well as a comprehension of the applications of probability involved. The assumed level for the exercises is high school graduate. The exercises are tailored to students that have a basic understanding of algebra with some ideas of probability. The materials required for the exercises are no more than a pair of dice and a calculator.