Isometries of Tone Structures

There is an undeniable connection between mathematics and music. Music is created by patterns forming a musical idea. The basics behind a musical idea are that a pitch defines a note that is used sometimes along with other notes to define a series of connected rhythms to create the desired effect. Therefore one can see the importance of pitches in composing most pieces of music. The goal of this worksheet is to provide a method of comparing pieces of music by determining if two compositions have isometric tone structures. This could be a useful tool in analyzing, comparing and contrasting pieces of music.

Pitches

Vibrations per second define pitch. Though human ears are not equipped to tell the difference between every pitch, there are infinitely many different pitches. There is a subset of pitches that form a basis for most compositions of music given below:

Pitch	Vibrations per Second
C	261.63
$C^{\#}$ or D_b	277.18
D	293.66
$D^{\#}$ or D_b	311.13
E	329.63
F	349.23
$F^{\#}$ or G_b	369.99
G	392.00
$G^{\#}$ or A_b	415.30
A	440.00
$A^{\#}$ or B_b	466.16
B	493.88

The above table gives the pitches of the octave originating at "middle C." When two notes are an octave apart, the pitch of the higher note is twice that of the pitch of the lower note. When this occurs, the notes are given the same name but the pitches are different!! This provides us with a set that forms a basis for our pitches

$$P_j \in \{261.63,\quad 277.18,\quad 293.66,\quad 311.13,\quad 329.63,\quad 349.23,\quad 369.99,\quad 392.00,\quad 415.30,\quad 440.00,\quad 466.16,\quad 493.88\}$$

Expressing Pitches in Set Notation

Define the set C, by $C = \{2^n (261.63) | n \in Z\}$. This set gives the pitches of all notes that are called C. Why?

Define the set that gives the pitches of all notes call $G^{\#}$ or A_b.

In general, we can write $N_i = \{2^n P_i | n \in Z\}$ where N_i gives the name of the family of pitches and P_i gives the corresponding pitch from the octave originating at middle C.

Putting a Metric on Pitches

A *metric* on a set X is a function d: XxX$\rightarrow \Re$ such that:

a) $d(x, y) \geq 0 \ \forall x, y \in X$
b) $d(x, y) = 0 \Leftrightarrow x = y$
c) $d(x, y) = d(y, x) \ \forall x, y \in X$
d) $d(x, y) + d(y, z) \geq d(x, z) \ \forall x, y, z \in X$

If d is a metric on a set x, then the ordered pair (x,d) is called a *metric space.*[1]

Our set X of pitches is the union of N_i meaning, X consists of all the pitches formed by the different notes within the octave of middle c (refer to the table above). Therefore, $x \in X \Rightarrow x = 2^{n_i} P_j$ where n_i is some integer.

$P_j \in \{261.63, \ 277.18, \ 293.66, \ 311.13, \ 329.63, \ 349.23, \ 369.99, \ 392.00, \ 415.30, \ 440.00, \ 466.16, \ 493.88\}$

P_j is one of the pitches in the octave of middle c.
Define a distance function, d, by $d(x, y) = |x - y|$.

To show that our set X and distance function d form a metric space, we need to show that each part of the definition is satisfied. This is left as an exercise.

a) $d(x, y) \geq 0 \ \forall x, y \in X$.

b) $d(x, y) = 0 \Leftrightarrow x = y$

c) $d(x, y) = d(y, x) \ \forall x, y \in X$

[1] patty

d) d(x, y) + d(y, z) $\geq$ d(x, z) $\forall$x, y, z $\in$ X

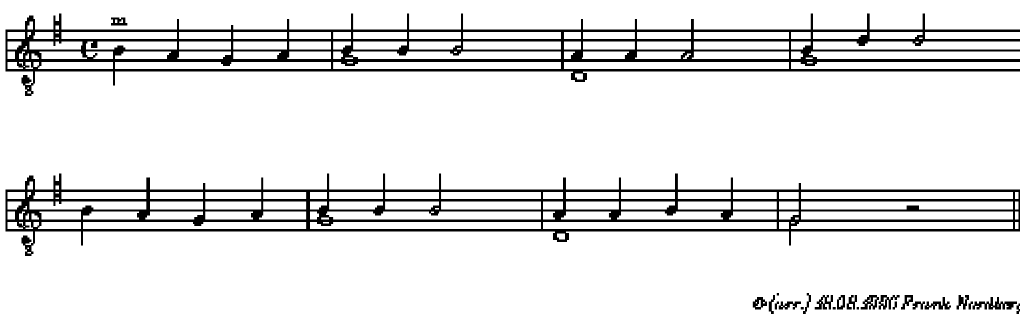
Also any subset of X forms a metric space with the distance function, $d(x, y) = |x - y|$.

Isometries

An *isometry* is a one-to-one mapping, f: (X, d) $\to$ (Y, d') of a metric space (X, d) onto (Y, d') such that distance is preserved. In other words for all x_1 and x_2 in X we have: d($x_{1,}$ x_2) = d'(f(x_1)$_,$ f(x_2)).

Two spaces are *isometric* if there exists an isometry from one space onto the other.[2]

Now consider the tune "Mary had a Little Lamb" as provided below.


[3]

For those who cannot read treble clef, the pitches of this tune are B, A, G, D, higher D and $F^{\#}$. All of these pitches except the higher D are within the middle octave. The higher D is within the first octave above the middle c octave which gives n=1. Let M_1 be the set of pitches for "Mary had a Little Lamb."
This gives $M_1 = \{493.88, \quad 440.00, \quad 392.00, \quad 2 \cdot 293.66, \quad 369.99, \quad 293.66\}$.

Now lets consider another version of Mary had a Little Lamb an octave above the version above.

The set of pitches for this version is:
$M_2 = \{2 \cdot 493.88, \quad 2 \cdot 440.00, \quad 2 \cdot 392.00, \quad 2^2 \cdot 293.66, \quad 2 \cdot 369.99, \quad 2 \cdot 293.66\}$

Now define $f : M_1 \to M_2$ by,

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[2] fgt
[3] http://www.free-sheet-music-for-piano-and-guitars.com/guitar.php

$$f = \begin{cases} 493.88 \rightarrow 2 \cdot 493.88 \\ 440.00 \rightarrow 2 \cdot 440.00 \\ 392.00 \rightarrow 2 \cdot 392.00 \\ 369.99 \rightarrow 2 \cdot 369.99 \\ 2 \cdot 293.66 \rightarrow 2^2 \cdot 293.66 \\ 293.66 \rightarrow 2 \cdot 293.66 \end{cases}$$

This function is obviously one-to-one and onto. Now show that f preserves distances.

This gives us an isometry, which implies that M_1 and M_2 are isometric.

Try to find another function that preserves distances or put a different metric on M_2 or M_1 by defining new distance functions and verifying that they form metric spaces as done above. Do not feel bad if you do not succeed!! Just remember you are playing around with pieces of a puzzle trying to make them fit. However, please do show your attempts.